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Nonnegative Sums of Roots of Unity

Morris Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(July 9, 1975)

Let q, n be integers > 1, and let $\rho_1, \rho_2, \ldots, \rho_n$ be distinct qth roots of unity. It is shown that $\rho_1^k + \rho_2^k + \ldots + \rho_n^k \geq 0$ for all integral k if and only if n is a divisor of q and the set $\{\rho_1, \rho_2, \ldots, \rho_n\}$ coincides with the set $\{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}$, where $\zeta_n = \exp(2\pi i/n)$.

Key words. Algebraic numbers; conjugates, roots of unity.

The theorem proved in this note has its origins in problems concerning the characterization of the spectrum of a nonnegative matrix which have been studied by S. Friedland (as yet unpublished). The question answered by the theorem was posed to the author by Dr. Friedland.

THEOREM: *Let* q,n *be integers* > 1, *and let* $\rho_1, \rho_2, \ldots, \rho_n$ *be distinct* qth *roots of unity. Then* $\rho_1^k + \rho_2^k + \ldots + \rho_n^k \geq 0$ *for all integral* k *if and only if* n *is a divisor of* q *and the set* $\{\rho_1, \rho_2, \ldots, \rho_n\}$ *coincides with the set*

$$\{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}, \text{ where } \zeta_n = \exp(2\pi i/n).$$

PROOF. We first assume the nonnegativity of the sums. Then after a suitable renumbering we may put $\rho_r = \zeta^{a_r}$, where $\zeta = \zeta_q = \exp(2\pi i/q)$ is a primitive qth root of unity, and

$$0 \leq a_1 < a_2 < \ldots < a_n \leq q-1. \tag{1}$$

Put

$$\alpha_k = \zeta^{a_1 k} + \zeta^{a_2 k} + \ldots + \zeta^{a_n k}.$$

Then by the hypotheses of the theorem,

$$\alpha_k \geq 0 \quad \text{for all integral } k. \tag{2}$$

It is of course sufficient to consider only those k such that $0 \leq k \leq q-1$, since $\alpha_{k+q} = \alpha_k$.

AMS Subject Classification: 12A35, 12A45.

We have

$$\sum_{k=0}^{q-1} \alpha_k = \sum_{r=1}^{n} \sum_{k=0}^{q-1} \zeta^{a_r k} = \begin{cases} 0, & a_1 \neq 0 \\ q, & a_1 = 0. \end{cases}$$

Thus if $a_1 \neq 0$, $\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1} = 0$, which implies by (2) that $\alpha_0 = \alpha_1 = \ldots = \alpha_{q-1} = 0$. This is not possible, since $\alpha_0 = n$. It follows that $a_1 = 0$, and that

$$\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1} = q. \tag{3}$$

The proof will be by induction on $\Omega(q)$, the total number of prime factors of q. Consider first $\Omega(q) = 1$, so that q is prime. Since q is prime, the algebraic integers $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ form a complete set of conjugates; and since an algebraic number is zero if and only if every one of its conjugates is zero, there are just two possibilities:

(a) $$\alpha_k \neq 0, \qquad 1 \leq k \leq q-1.$$

(b) $$\alpha_k = 0, \qquad 1 \leq k \leq q-1.$$

Assume first that (a) holds. Then (2) implies that $\alpha_k > 0$, $1 \leq k \leq q-1$. We have from (3) that

$$1 = (\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1})/q \geq (\alpha_0 \alpha_1 \ldots \alpha_{q-1})^{1/q},$$

$$\alpha_0 \alpha_1 \ldots \alpha_{q-1} \leq 1,$$

$$\alpha_1 \alpha_2 \ldots \alpha_{q-1} < 1,$$

since $\alpha_0 = n < 1$. It follows that $0 < \alpha_1 \alpha_2 \ldots \alpha_{q-1} < 1$, which is a contradiction, since $\alpha_1 \alpha_2 \ldots \alpha_{q-1}$ is the norm of the algebraic integer α_1 and so a rational integer.

Now assume that (b) holds, and that $n \leq q-1$. Then we must have that

$$\alpha_k = 0, \qquad 1 \leq k \leq n.$$

Thus

$$\sum_{r=1}^{n} \zeta^{a_r k} = 0, \qquad 1 \leq k \leq n. \tag{4}$$

Because of (1), the matrix $S = (\zeta^{a_r k})$, $1 \leq r, k \leq n$, must be non-singular, which contradicts (4). Thus the assumption $n \leq q-1$ is not possible, and so $n = q$, and the integers a_r must satisfy $a_r = r-1$, $1 \leq r \leq n$. Thus the result follows in this case.

Now suppose the result proved for all q such that $\Omega(q) < N$, $N \geq 2$, and let q be any positive integer such that $\Omega(q) = N$. Let t be any integer such that $1 \leq t \leq n$. We have

$$\alpha_k = \sum_{r=1}^{n} \zeta^{a_r k},$$

$$\zeta^{-a_t k} \alpha_k = \sum_{r=1}^{n} \zeta^{(a_r - a_t)k},$$

$$\sum_{k=0}^{q-1} \zeta^{-a_t k} \alpha_k = \sum_{r=1}^{n} \sum_{k=0}^{q-1} \zeta^{(a_r - a_t)k} = q,$$

sine (1) implies that $a_r - a_t \equiv 0 \bmod q$ if and only if $r = t$.
Thus

$$\sum_{k=0}^{q-1} \zeta^{-a_t k} \alpha_k = \sum_{k=0}^{q-1} a_k = q,$$

$$\sum_{k=1}^{q-1} (1 - \zeta^{-a_t k})\ \alpha_k = 0. \tag{5}$$

Now use the fact that α_k is real, take the complex conjugate of both sides of (5), and add the resulting equations. The result is that

$$\sum_{k=0}^{q-1} (2 - \zeta^{a_t k} - \zeta^{-a_t k})\ \alpha_k = 0. \tag{6}$$

Since $2 - \zeta^{a_t k} - \zeta^{-a_t k} \geqslant 0$, and $\alpha_k \geqslant 0$ by assumption, we conclude from (6) that

$$(2 - \zeta^{a_t k} - \zeta^{-a_t k})\alpha_k = 0;$$

and since

$$2 - \zeta^{a_t k} - \zeta^{-a_t k} = -\zeta^{-a_t k}\ (1 - \zeta^{a_t k})^2,$$

we may also conclude that

$$(1 - \zeta^{a_t k})\alpha_k = 0, \qquad 1 \leqslant t \leqslant n, \qquad \text{all integral } k. \tag{7}$$

Now the set $\{k : 1 \leqslant k \leqslant q\}$ coincides with the set $\{ql/d\colon d|q,\ (l, d) = 1\}$, where the notation means that d runs over the positive divisors of q and l runs over the positive integers $\leqslant d$ and relatively prime to d.

Suppose first that there is no value of $d > 1$ such that d divides a_t for all t. Then (7) and the decomposition above easily imply that $\alpha_k = 0$ for $1 \leqslant k \leqslant q-1$ (here we must use the fact that the conjugates of ζ_d are precisely ζ_d^l, $(l, d) = 1$, so that $\alpha_{q/d}$ vanishes if and only if $\alpha_{ql/d}$ vanishes). Then it follows as before that $n = q$, and that $a_r = r - 1$, $1 \leqslant r \leqslant n$. Next suppose that there is a $d > 1$ such that d divides a_t for all t. Put $a_t = db_t$, $1 \leqslant t \leqslant n$.

Then

$$\alpha_k = \sum_{r=1}^{n} \zeta_q^{a_r k} = \sum_{k=1}^{n} \zeta_{q/d}^{b_r k} \geqslant 0 \text{ for all integral } k,$$

and

$$0 = b_1 < b_2 < \ldots < b_n \leqslant q/d - 1.$$

Since $\Omega(q/d) < N$, the induction hypothesis implies that n divides q/d, and that

$$b_r = q(r-1)/nd, \qquad 1 \leqslant r \leqslant n.$$

It follows that n divides q and that

$$a_r = db_r = q(r-1)/nd, \qquad 1 \leq r \leq n.$$

Hence the result holds for any q such that $\Omega(q) = N$, and the proof (in this direction) is complete.

Finally, if n divides q and $a_r = q(r-1)/n$, $1 \leq r \leq n$, then

$$\alpha_k = \sum_{r=1}^{n} \zeta^{q(r-1)/n} = \sum_{r=1}^{n} \zeta_n^{(r-1)k} = \begin{cases} n, & n/k \\ 0, & \text{otherwise} \end{cases}$$

so that $\alpha_k \geqslant 0$ for all k.

This completes the proof of the theorem.

(Paper 80B1–430)

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Plane-Wave Scattering-Matrix Theory of Antennas and Antenna-Antenna Interactions: Formulation and Applications

David M. Kerns

Institute for Basic Standards, National Bureau of Standards, Boulder, Colorado 80302

(September 29, 1975)

In recent years a considerable amount of theoretical, experimental, and computational work in the development and application of techniques for accurate measurement of microwave antennas has been successfully completed at the National Bureau of Standards (and work is continuing). This paper presents and extends the basic plane-wave scattering-matrix formalism and presents new generalized or adjoint reciprocity relations for antennas. The PWSM formalism is eminently suitable for the formulation and solution of problems involving interactions at arbitrary distances and for the expression of conventional asymptotic quantities, such as gain, effective area, and polarization. It has in particular enabled derivation of two new techniques that permit accurate, "probe-corrected" antenna measurements at greatly reduced distances: (1) by deconvolution of transverse scanning data, taken with $d \ll d_R$ (where $d_R = a^2/2\lambda$) and (2) by extrapolation of received signal observed as a function of distance d, with $d \sim d_R$. These techniques basically determine the scalar product, C, of two vectors characteristic respectively of the transmitting and the receiving antennas. Formulas for utilization of C-data, taking full account of polarization characteristics and not requiring reciprocal antennas, are given for (a) one-unknown-antenna, (b) generalized two-identical-antenna, and (c) generalized three-antenna measurement techniques.

Key words: Antenna-antenna interactions; antenna measurements; antenna theory; scattering-matrix theory of antennas.

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Some Conventions and Notation Used Consistently

(a) *Conventions*

1. Complex numbers are sometimes called phasors; they are not called vectors.

2. Bold-face symbols denote vectors or dyadics defined in "ordinary" space or in wavenumber space. Components may be complex numbers.
3. Scalar and vector products of two vectors are denoted by $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$, respectively. The scalar product of three vectors taken in the cyclic order **A, B, C** is denoted by **[ABC]**.
4. A superposed bar denotes the complex conjugate.
5. The magnitude of a complex number z is denoted by $|z|$.
6. The squared magnitude of a vector **V** is defined by $\mathbf{V} \cdot \overline{\mathbf{V}}$ and denoted by $|\mathbf{V}|^2$.
7. The "square" of a vector **V** is defined by $\mathbf{V} \cdot \mathbf{V}$ and denoted by V^2. Example: $k^2 = \omega^2 \mu \epsilon$ (see the list of symbols following).
8. "Transverse" means perpendicular to the z axis unless otherwise indicated.
9. "On-axis" refers to the z axis of coordinates, not to an axis possibly suggested by antenna geometry.
10. The (suppressed) time dependence is exp $(-i\omega t)$.

(b) *Roman Letters*

$\mathbf{a}_q$: Complete vectorial spectrum for **E** of incident field (p. 14).
$\mathbf{A}_q$: Transverse part of $\mathbf{a}_q$ (p. 15).
a_0: Incident wave-amplitude in antenna feed transmission line or waveguide (p. 9).
$\mathbf{b}_q$: Complete vectorial spectrum for **E** of scattered or radiated field (p. 14).
$\mathbf{B}_q$: Transverse part of $\mathbf{b}_q$ (p. 15).
b_0: Emergent wave-amplitude in antenna feed transmission line or waveguide.
$d\mathbf{K}$: Symbolizes surface element in double integrals in k_x, k_y space.
$d\mathbf{R}$: Symbolizes surface element in double integrals in x, y space.
E: "Electric field" (complex representation).
$\mathbf{E}_t$: Transverse part of **E**.
$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$: Fixed, orthogonal, right-handed system of unit vectors.
$\mathbf{e}_{||}, \mathbf{e}_\perp, \mathbf{e}_k$: Orthogonal, right-handed system of unit vectors tied to **k** (p. 13).
$G_q(\mathbf{K})$: Power-gain function evaluated in the direction of **k**; $q = 1$ or 2 implies $\mathbf{k} = \mathbf{k}^+$ or $\mathbf{k}^-$, respectively (p. 23).
H: "Magnetic field" (complex representation).
$\mathbf{H}_t$: Transverse part of **H**.
i: $\sqrt{-1}$.
k: Propagation vector; components k_x, k_y, k_z.
$\mathbf{k}^\pm$: Propagation vector with z-component equal to $\pm\gamma$.
K: Transverse part of **k** (**K** is chosen real in this work).
k: $\sqrt{\mathbf{k} \cdot \mathbf{k}} = \omega\sqrt{\mu\epsilon}$ (a real quantity in this work).
K: $\sqrt{\mathbf{K} \cdot \mathbf{K}}$
m, n: Index taking on values 1, 2 and indicating association with unit vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$(p.).
p, q: Index taking on values 1, 2 and indicating association with regions to the "right" and to the "left" of an antenna or scatterer (p. 12).
r: Position vector (a real vector); components x, y, z.
R: Transverse part of **r**.
r: Magnitude of **r**.
R: Magnitude of **R**.
$\mathbf{S}_{0q}$: Transverse vectorial receiving characteristic (p. 17).
$\mathbf{s}_{0q}$: Complementary receiving characteristic (p. 25).
$\mathbf{S}_{q0}$: Transverse vectorial transmitting characteristic (p. 16).
$\mathbf{s}_{q0}$: Complete transmitting characteristic (p. 23).
$w_q(\mathbf{K})$: Polarization index for incident plane-waves (p. 25).
Y_0: $\sqrt{\epsilon/\mu}$, wave admittance for simple plane-waves in medium with parameters ϵ, μ.

(c) *Greek Letters*

γ: $\sqrt{k^2 - K^2}$, taken positive when $K < k$, positive imaginary when $K > k$; $k_z = \pm\gamma$ (p.).
$\delta(k_x)$: Dirac delta "function."

$\delta(\mathbf{K})$: Abbreviation for $\delta(k_x)\delta(k_y)$.

ϵ, ϵ_0: Permittivity of homogeneous, isotropic, dissipationless medium.

η_0: Characteristic admittance for the propagated mode in waveguide feed (p. 9).

η_1: z-component of wave-admittance for TM plane-waves in space; $\eta_1 = \omega\epsilon/\gamma = Y_0 k/\gamma$, and for $0 \leqslant \theta \leqslant \pi$, $= Y_0/|\cos\theta|$ (p. 12).

η_2: z-component of wave-admittance for TE plane-waves in space; $\eta_2 = \gamma/(\omega\mu) = Y_0\gamma/k$, and for $0 \leqslant \theta \leqslant \pi$, $= Y_0|\cos\theta|$ (p. 12).

θ: Polar angle in spherical polar coordinates (p. 13).

$\boldsymbol{\kappa}_1$: Unit vector $\equiv \mathbf{K}/K$ (p. 11).

$\boldsymbol{\kappa}_2$: Unit vector $\equiv \mathbf{e}_z \times \boldsymbol{\kappa}_1$ (p. 11).

μ, μ_0: Permeability of homogeneous, isotropic, dissipationless medium.

ρ_{0q}: Polarization index for receiving characteristics (p. 25).

ρ_{q0}: Polarization index for transmitting characteristics (p. 24).

$\sigma_q(\mathbf{K})$: Effective area for reception (p. 24); $q = 1$ or 2 implies $\mathbf{k} = \mathbf{k}^-$ or $\mathbf{k} = \mathbf{k}^+$, respectively.

ϕ: Azimuthal angle of plane or spherical polar coordinates (p. 13).

ω: Angular velocity as in the suppressed time factor $\exp(-i\omega t)$.

Introduction

In section 1 we give, quite thoroughly as far as definitions and notation are concerned, a formulation of the plane-wave scattering-matrix for antennas and scatterers.[1] In previous publications [1, 2],[2] only a "one-side" matrix description of antennas, using one spatial reference plane, was given explicitly. To provide a more complete exposition, we give here the formulation of the complete scattering matrix obtained by enclosing the antenna between two planes and considering incident and emergent plane waves on both planes.

The plane-wave scattering matrix is put forth as a good basis for advanced antenna theory, especially such theory related to antenna measurement techniques. While the results given may be persuasive, numerous analytical examples and applications which illustrate and extend the theory are available—and are not included. These are intended for a more comprehensive publication. Apart from appendices A and B, relatively little theory is developed in this paper. However, logical interrelations are indicated, and more detail is given when it is a question of relating new concepts to more familiar concepts and to practicable antenna measurement techniques.

An "acoustics translation" of much of the present work has been published [3]. Because of the relative simplicity of the acoustic wave-fields involved, this may be found helpful in illuminating the electromagnetic case.

The scattering matrix formulation properly includes basic expressions for power transfer and for reciprocity.

The reciprocity relations are stated in a generalized form, using the concept of mutually adjoint antennas. The concept of generalized reciprocity is not in itself new [4, 5], but apparently it has not previously been formulated for antennas and scatterers (see, however, remark following eq. (1.6–21)). The generalized or *adjoint* reciprocity relations have found substantial application in research establishing the foundations of the extrapolation technique [6, 7]. They are used in this paper in the formulation of possible new antenna measurement techniques, predicated upon the physical realization of mutually adjoint antennas.

In section 2 the plane-wave scattering matrix approach is used to obtain a complete and general solution to the problem of coupled antennas. In spite of their formal appearance, the general results obtained represent the heart of the present theory, and provide a fruitful and reliable basis for additional results (including those reported here). Two of these results are the deconvolution and extrapolation techniques, labeled (1) and (2) in the Abstract and outlined in section 3. These techni-

[1] Since an antenna is in general a scattering object, and a passive antenna externally is merely a scattering object, the consideration of scattering is included in the full consideration of antennas and need not always be mentioned explicitly.

[2] Figures in brackets indicate the literature references at the end of this paper.

ques basically determine values of the scalar product, called the *coupling product*, of two two-component vectors characteristic of the two antennas involved. (This statement also applies to the convetional far-field antenna measurement methods, provided due attention is paid to polarization characteristics.)

Utilization of coupling-product data is discussed as a separate topic. Here the analytical problem is primarily geometric and algebraic, with the exact form depending upon what is considered known a priori and what information is sought. Three classes of antenna measurement situations [labeled (a), (b), and (c) in the Abstract] are discussed in section 4. The order of listing and discussion is roughly that of decreasing a priori information and increasing complexity.

An increasing body of experimental results involving various combinations of the several techniques identified above may be found in the literature [8–11, 54]. Moreover, the error analysis required for determination of accuracy in concrete measurement situations is approaching completion (Kanda [12], Yaghjian [13], Newell [14]).

The combination of the techniques labeled (1) and (a) amounts to a technique for correction of near-field antenna measurements made with an arbitrary but known measuring antenna. The ability to obtain the true radiated spectrum of an unknown antenna, fully corrected for the effects of the measuring antenna, incidentally implies the ability to obtain corresponding true values of **E** and **H** in the near field, similarly fully corrected. Frequently, and in particular in the following paragraphs, the measuring antenna will be referred to as a "probe."

The general subject of determination of far-field antenna patterns from near field data is surveyed and an extensive bibliography is given in a recent paper by Johnson et al. [15]. This paper should be consulted for an overview of the subject. Here we mention specifically only certain earlier work in which the "probe-correction problem" was considered or which represented steps leading to the eventual simple, rigorous, and general deconvolution solution of the problem.

Woonton, in 1953 [16], obtained an integral expression for the near-field response of a linear (= thin wire) antenna and discussed probe effects qualitatively. Woonton stated that the problem had not been critically discussed previously. Dayhoff (1956) [17], using scalar waves, plane-wave spectrum analysis, and reciprocity, introduced a version of the very important *transmission integral.* (Dayhoff used the transmission integral to obtain an approximate solution of the diffraction correction problem in microwave interferometry. A rigorous and more general version of this solution was presented by Kerns in 1957 [8].) Brown [19] (1958), using plane-wave spectrum analysis and reciprocity [20], obtained a version of the transmission integral and used it to give an approximate analysis of probe effects, assuming simple, known data for both antennas involved. More work along this line (limited to two-dimensions) was done by Jull [21, 22]. In 1961 Brown and Jull [23] gave a rigorous and general solution to the probe correction problem for the two-dimensional case using cylindrical wave functions. The use of two-dimensional solutions in certain three-dimensional problems has been suggested, and was studied experimentally by Martin [24], but is not valid for any three-dimensional problems. The proper extension to three dimensions in spherical or cylindrical coordinates is far from trivial. See Jensen [25], Leach and Paris [26], and Wacker [27].

The key to the present solution to the probe-correction problem is the use of plane-wave analysis and rectangular coordinates and the recognition that planar scanning would permit solution of the integral equation presented by the transmission integral by Fourier inversion[3] (better called deconvolution in the existing context). This solution was presented in 1963 by Kerns [28], and again, including application of a two-dimensional, spatial sampling theorem, in 1967, at a University of Colorado Advanced Electromagnetic Theory Summer Course. Archival publication, accompanied by substantial experimental application and verification by Baird et al. [8], was accomplished in 1970 [2]. The transmission integral used, though similar to that derived by Brown, was actually obtained by methods used in Kerns and Dayhoff [1], where it was derived without recourse to reciprocity and was explicitly recognized as the first term in an infinite series of interaction terms. (See the remarks at the end of appendix A, below.)

[3] These matters are adequately discussed later in this paper.

The Kerns and Dayhoff paper, not originally considered to be in the domain of antenna theory by its authors, has served as an important base for much of the material embodied in the present and related papers and in other work as yet unpublished.

1. Plane-Wave Scattering-Matrix for Antennas and Scatterers

1.1. Representation of Fields on S_0, Definition of a_0 and b_0

Let us consider the antenna system shown schematically in figure 1. We choose a (mathematical) terminal surface S_0 in the waveguide feed and define a supplementary surface S_a, such that $S_a + S_0$ forms a closed surface enclosing the source or detector associated with the antenna. The surface S_a coincides with shielding, which is required to make the problem well defined (both experimentally and theoretically!). As an important measure of simplicity, we consider only the case of a single waveguide feed, supporting just one propagated mode. We employ conventional phasor wave amplitudes a_0 and b_0 for the incident and emergent traveling wave components at S_0. These wave amplitudes are fully defined by the following four equations.

The tangential (= transverse) components of $\mathbf{E}$ and $\mathbf{H}$ on S_0 are given by

$$\mathbf{E}_{0t} = (a_0 + b_0)\mathbf{e}_0(\mathbf{r}), \tag{1.1-1a}$$

$$\mathbf{H}_{0t} = \eta_0(a_0 - b_0)\mathbf{h}_0(\mathbf{r}), \quad (\mathbf{r} \text{ on } S_0) \tag{1.1-1b}$$

where $\mathbf{e}_0(\mathbf{r})$ and $\mathbf{h}_0(\mathbf{r})$ are real basis fields for the mode involved, subject to the impedance normalization

$$\mathbf{h}_0(\mathbf{r}) = \eta_0^{-1}\eta_w \mathbf{n}_0 \times \mathbf{e}_0(\mathbf{r}) \tag{1.1-2}$$

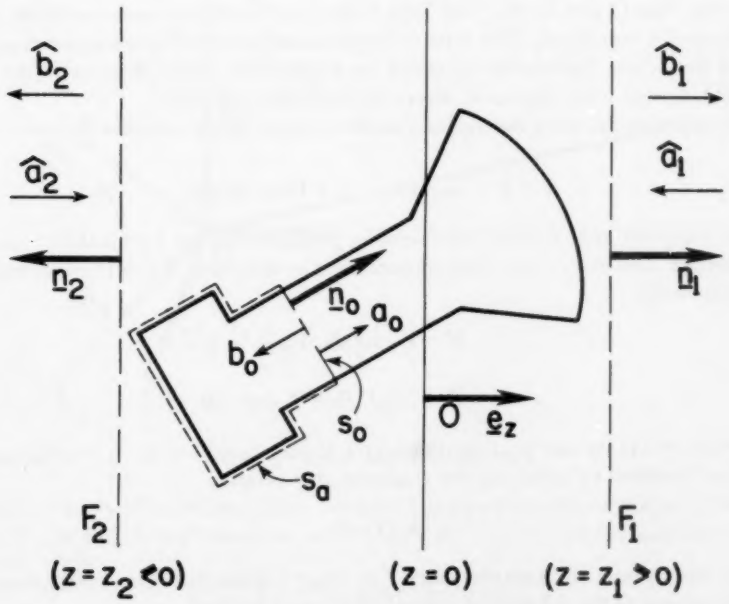

FIGURE 1. *Some notation for plane-wave scattering-matrix description of antennas.*

Arrows indicate the association of a's and b's with incident and emergent waves respectively. Antenna-system representation is symbolic: no particular size, shape, symmetry, type, orientation, or position is implied. In any concrete case, position and orientation of antenna relative to coordinate system must be established, as with the aid of fiducial marks on the antenna.

and to the power normalization

$$\int_{S_0} [\mathbf{e}_0 \mathbf{h}_0 \mathbf{n}_0]\, dS = 1. \tag{1.1-3}$$

Here the integrand is the scalar triple product; $\mathbf{n}_0$ is the unit normal vector, drawn inward with respect to the antenna; η_0 is the characteristic admittance, and η_w is the wave admittance for the mode involved. Equation (3) establishes peak-value normalization for a_0 and b_0, so that net time-average power input to the antenna at S_0 is given by

$$P_0 = \frac{1}{2} \operatorname{Re} \int_{S_0} \mathbf{E}_{0t} \times \overline{\mathbf{H}}_{0t} \cdot \mathbf{n}_0 dS = \frac{1}{2} \eta_0 (|a_0|^2 - |b_0|^2), \tag{1.1-4}$$

where Re denotes that the real part is to be taken, the superposed bar denotes the complex conjugate, and the vertical bars denote absolute values. Remarks: (a) The impedance normalization shown above is more flexible than that used previously [1, 2]. It allows one to choose whatever characteristic impedance or admittance is deemed familiar or convenient for purposes of measurement-related calculations. Examples are, for waveguide, $\eta_0 = 1$ or $\eta_0 = \eta_w$; and, for coaxial line, the conventional characteristic admittance ($= 2\pi\sqrt{\epsilon/\mu}/\ln(b/a)$ in conventional notation). (b) Power normalization differs slightly from that used previously: Powers of the factor (2π) have been redistributed. (c) For a detailed discussion of the material in this subsection and for the small but essential amount of microwave network theory needed in measurement-related calculations, see [29] or the especially prepared report [30]. For more elementary discussion see also [31].

1.2. Representation of Fields in Space; Definition of $a_q(m, K)$ and $b_q(m, K)$

We choose a rectangular coordinate system $0xyz$ (with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$) so that the considered antenna system may be confined entirely to the space between the (mathematical) surfaces F_1 and F_2 at $z = z_1 > 0$ and at $z = z_2 < 0$, as suggested in figure 1. The electromagnetic fields in the regions to the "right" and to the "left" are to be represented as superpositions of plane-wave solutions of Maxwell's equations. This type of representation is well known (see e.g., [32]), at least for solutions of the scalar Helmholtz equation; an appropriate generalization to the (vector) electromagnetic field, though often shunned, offers no particular difficulty.

The electromagnetic field in the regions under consideration satisfies Maxwell's equations in the form

$$\nabla \times \mathbf{E} = i\omega\mu \mathbf{H}, \qquad \nabla \times \mathbf{H} = -i\omega\epsilon \mathbf{E}, \tag{1.2-1}$$

where μ, ϵ are constant real scalars representing respectively the permeability and the permittivity of the medium, and $\exp(-i\omega t)$ time dependence is assumed. We derive our basis fields from the general plane wave

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{T} \exp(i\mathbf{k} \cdot \mathbf{r}), \\ \mathbf{H} &= (\omega\mu)^{-1} \mathbf{k} \times \mathbf{T} \exp(i\mathbf{k} \cdot \mathbf{r}), \end{aligned}\right\} \tag{1.2-2a}$$

which is a solution of (1) for any propagation vector $\mathbf{k}$ such that $k^2 \equiv \mathbf{k} \cdot \mathbf{k} = \omega^2\mu\epsilon$ and any vector $\mathbf{T}$ (independent of position $\mathbf{r}$) satisfying the transversality relation

$$\mathbf{k} \cdot \mathbf{T} = 0. \tag{1.2-2b}$$

In spite of this occurrence of "transversality," in what follows the term "transverse" will mean transverse with respect to the z direction unless otherwise specified.

The propagation vector will be regarded as a function of its transverse components k_x, k_y (which are chosen real); the z-component is thus

$$k_z = \pm\gamma, \tag{1.2-3a}$$

where $\gamma^2 = k^2 - k_x^2 - k_y^2$. It will be convenient to denote the transverse part of the propagation vector by $\mathbf{K}$, so that $\mathbf{K} = k_x \mathbf{e}_x + k_y \mathbf{e}_y$ and

$$\gamma = (k^2 - K^2)^{1/2}. \tag{1.2-3b}$$

Since k_x, k_y must be allowed to vary independently in the range $(-\infty, \infty)$, real and imaginary values of γ will occur. γ will be taken positive for $K^2 < k^2$, positive imaginary for $K^2 > k^2$. Superscripts "+" and "−" will be used when it is desired to indicate the choice of sign associated with k_z. When γ is real, the exponentials $\exp(i\mathbf{k}^+ \cdot \mathbf{r})$ and $\exp(i\mathbf{k}^- \cdot \mathbf{r})$ respectively represent simple plane waves traveling into the $+z$ and $-z$ hemispheres. When γ is imaginary, the exponentials represent inhomogeneous plane waves with propagation of phase in the transverse directions and exponential attenuation of amplitude ("evanescence") in the + and − z directions, respectively.

In virtue of the relation $\mathbf{k} \cdot \mathbf{T} = 0$, (2a) yields just two linearly independent fields, hence just two basis fields, for any given $\mathbf{k}$. The appropriate polarizations for the basis fields are those with the electric vectors parallel or perpendicular to the plane of $\mathbf{k}$ and $\mathbf{e}_z$, which is the plane of incidence for a wave incident on any plane $z = \text{const}$. This choice of polarizations yields "transverse magnetic" and "transverse electric" waves; the same choice of polarizations, usually labelled "$E_{||}$" and "$E_{\perp}$", simplifies the derivation of Fresnel's equations in optics or electromagnetic theory.

In order to set up the basis fields in the desired form, we require the *transverse* unit vectors

$$\boldsymbol{\kappa}_1 = \mathbf{K}/K, \quad \boldsymbol{\kappa}_2 = \mathbf{e}_z \times \boldsymbol{\kappa}_1, \tag{1.2-4}$$

which are respectively in and perpendicular to the plane of $\mathbf{k}$ and $\mathbf{e}_z$. This part of the notation is illustrated in figure 2; $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ may be identified as radial and tangential unit vectors, as often

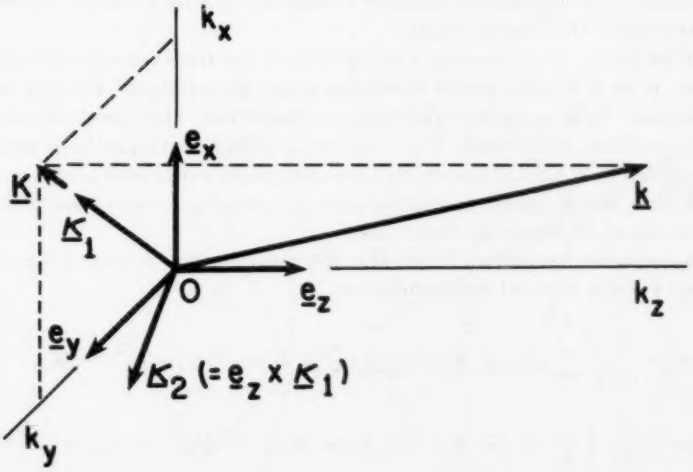

FIGURE 2. *Illustrating* $\mathbf{k}$, $\mathbf{K}$, $\boldsymbol{\kappa}_1$, *and* $\boldsymbol{\kappa}_2$.

associated with polar coordinates in a plane. As a temporary abbreviation[4] we put $u^{\pm} = \exp(i\mathbf{k}^{\pm} \cdot \mathbf{r})/(2\pi)$. For the $E_{||}$ (or TM) components we put $\mathbf{T} = \boldsymbol{\kappa}_1 \mp K\gamma^{-1}\mathbf{e}_z$ and obtain from (2a)

$$\left.\begin{aligned} \mathbf{E}_{\mathrm{I}}^{\pm} &= [\boldsymbol{\kappa}_1 \mp K\gamma^{-1}\mathbf{e}_z]u^{\pm}, \\ \mathbf{H}_{\mathrm{I}}^{\pm} &= \pm \eta_1 \mathbf{e}_z \times \boldsymbol{\kappa}_1 u^{\pm}, \end{aligned}\right\} \tag{1.2-5}$$

[4] The factor $1/(2\pi)$ in the definition of u represents a change in normalization consistent with that noted for a_0 and b_0 in the preceding subsection.

where $\eta_1 = \omega\epsilon/\gamma$. For the $E_\perp$ (or TE) components we take $\mathbf{T} = \boldsymbol{\kappa}_2$ and obtain from (2a)

$$\left.\begin{aligned} \mathbf{E}_{\frac{2}{2}}^{\pm} &= \boldsymbol{\kappa}_2 u^{\pm}, \\ \mathbf{H}_{\frac{2}{2}}^{\pm} &= [\pm\eta_2 \mathbf{e}_z \times \boldsymbol{\kappa}_2 + K(\omega\mu)^{-1}\mathbf{e}_z] u^{\pm}, \end{aligned}\right\} \tag{1.2-6}$$

where $\eta_2 = \gamma/(\omega\mu)$. Among other similarities it may be observed that η_1, η_2 are wave-admittances that correspond closely to the wave-admittances encountered in the theory of rectangular waveguide. Equations (5) and (6) furnish the desired basis fields; somewhat arbitrarily, we have chosen to make the expressions for transverse $\mathbf{E}$ as simple as possible. The normalization and orthogonality properties of the basis fields are of course implicit in the expressions themselves. (This is an interesting contrast to (1.1-1), where the field patterns are implicit and the normalizations explicit.)

Let us now examine the plane-wave representations of electromagnetic fields in the regions $z \geq z_1 > 0$ and $z \leq z_2 < 0$. We write

$$\mathbf{E}_q(\mathbf{r}) = \int_{-\infty}^{\infty}\int \sum_{m=1}^{2} [b_q(m, \mathbf{K})\mathbf{E}_m^{\pm}(\mathbf{K}, \mathbf{r}) + a_q(m, \mathbf{K})\mathbf{E}_m^{\mp}(\mathbf{K}, \mathbf{r})]\, dk_x dk_y, \tag{1.2-7a}$$

$$\mathbf{H}_q(\mathbf{r}) = \int_{-\infty}^{\infty}\int \sum_{m=1}^{2} [b_q(m, \mathbf{K})\mathbf{H}_m^{\pm}(\mathbf{K}, \mathbf{r}) + a_q(m, \mathbf{K})\mathbf{H}_m^{\mp}(\mathbf{K}, \mathbf{r})]\, dk_x dk_y. \tag{1.2-7b}$$

Here the index q takes on the values 1 and 2 and identifies quantities respectively associated with the regions to the "right" and to the "left" of the system considered; the upper and the lower superscript signs are associated with $q = 1$ and $q = 2$, respectively; and $b_q(m, \mathbf{K})$ and $a_q(m, \mathbf{K})$ are scalar spectral-density functions for outgoing and incoming waves, respectively. The electromagnetic fields given by (7) will satisfy Maxwell's equations provided that the necessary differentiations can be taken under the integral signs.

Now, as will be shown in a moment, a knowledge of the transverse components of $\mathbf{E}_q$ and $\mathbf{H}_q$ (in a single plane, in fact) is sufficient to determine $a_q(m, \mathbf{K})$ and $b_q(m, \mathbf{K})$; and hence, by (7), the entire electromagnetic field in each of the regions considered. The z-components of the fields in (7) are, strictly speaking, redundant. The transverse components are both necessary and sufficient for the expression of normal energy-flux and continuity conditions across a transverse plane. We find, in fact, that the inclusion of z-components is sometimes convenient and sometimes not. The following equations illustrate the latter case.

For the transverse components of $\mathbf{E}$ and $\mathbf{H}$ in the regions $z \geq z_1 \geq 0$ and $z \leq z_2 \leq 0$, we find from (5), (6) and (7) the Fourier integral representations

$$\mathbf{E}_{qt}(\mathbf{r}) = \frac{1}{2\pi}\int \sum_{m} [b_q(m, \mathbf{K})e^{i\gamma|z|} + a_q(m, \mathbf{K})e^{-i\gamma|z|}]\boldsymbol{\kappa}_m e^{i\mathbf{K}\cdot\mathbf{R}}\, d\mathbf{K}, \tag{1.2-8a}$$

$$\mathbf{H}_{qt}(\mathbf{r}) = \frac{1}{2\pi}\int \sum_{m} [b_q(m, \mathbf{K})\, e^{i\gamma|z|}\ a_q(m, \mathbf{K})e^{-i\gamma|z|}]\mathbf{n}_q \times \boldsymbol{\kappa}_m \eta_m(K) e^{i\mathbf{K}\cdot\mathbf{R}} d\mathbf{K}, \tag{1.2-8b}$$

where $\eta_1 = \omega\epsilon/\gamma$, $\eta_2 = \gamma/(\omega\mu)$, and $\mathbf{r} = \mathbf{R} + z\mathbf{e}_z$. Here and in subsequent expressions of this type summation over the values 1 and 2 of the polarization index m and integration over the infinite k_x, k_y plane is to be understood. The role of the index q is as described under (7); the use of $|z|$ and $-|z|$ assures proper phase variation for outgoing and incoming waves, respectively; and $\mathbf{n}_q$ is the outward normal unit vector on F_q: $\mathbf{n}_1 = \mathbf{e}_z$, $\mathbf{n}_2 = -\mathbf{e}_z$ (i.e., "outward" means with respect to the slab between the surfaces F_1 and F_2). The functions $a_q(m, \mathbf{K})$ and $b_q(m, \mathbf{K})$ may be regarded as modal terminal variables for the continuous spectrum; a mode is identified by a triplet of values (m, k_x, k_y) and a direction (rightward or leftward).

Explicit expressions for the spectral density functions $b_q(m, \mathbf{K})$ and $a_q(m, \mathbf{K})$ may be found from the Fourier inversion of the above equations:

$$b_q(m, \mathbf{K}) = \frac{e^{-i\gamma|z|}}{4\pi} \boldsymbol{\kappa}_m \cdot \int [\mathbf{E}_q(\mathbf{R}, z) + \eta_m^{-1} \mathbf{H}_q(\mathbf{R}, z) \times \mathbf{n}_q] e^{-i\mathbf{K}\cdot\mathbf{R}} d\mathbf{R}, \tag{1.2-9a}$$

$$a_q(m, \mathbf{K}) = \frac{e^{i\gamma|z|}}{4\pi} \boldsymbol{\kappa}_m \cdot \int [\mathbf{E}_q(\mathbf{R}, z) - \eta_m^{-1} \mathbf{H}_q(\mathbf{R}, z) \times \mathbf{n}_q] e^{-i\mathbf{K}\cdot\mathbf{R}} d\mathbf{R}. \tag{1.2-9b}$$

Here the integrations are to be taken over the entire x, y plane—as required by the Fourier inversion—for any suitable fixed value of z (for $q = 1$, $z \geq z_1$; for $q = 2$, $z \leq z_2$).

The spectral functions $b_q(m, \mathbf{K})$ and $a_q(m, \mathbf{K})$ are independent of z, although this may not be immediately apparent in (9), and the values of the functions are referred to the plane $z = 0$. If desired, the phases and amplitudes could be referred to other reference planes S_q at $z = Z_q$, say. However, the choice implicitly made, $Z_1 = Z_2 = 0$, is convenient at least for present purposes.

It is convenient at this point to introduce a number of definitions for future reference when and as needed.

We observe that (7a) can be written

$$\mathbf{E}_q(\mathbf{r}) = \frac{1}{2\pi} \int [\mathbf{b}_q(\mathbf{K}) e^{i\gamma|z|} + \mathbf{a}_q(\mathbf{K}) e^{-i\gamma|z|}] e^{i\mathbf{K}\cdot\mathbf{R}} d\mathbf{K}. \tag{1.2-10}$$

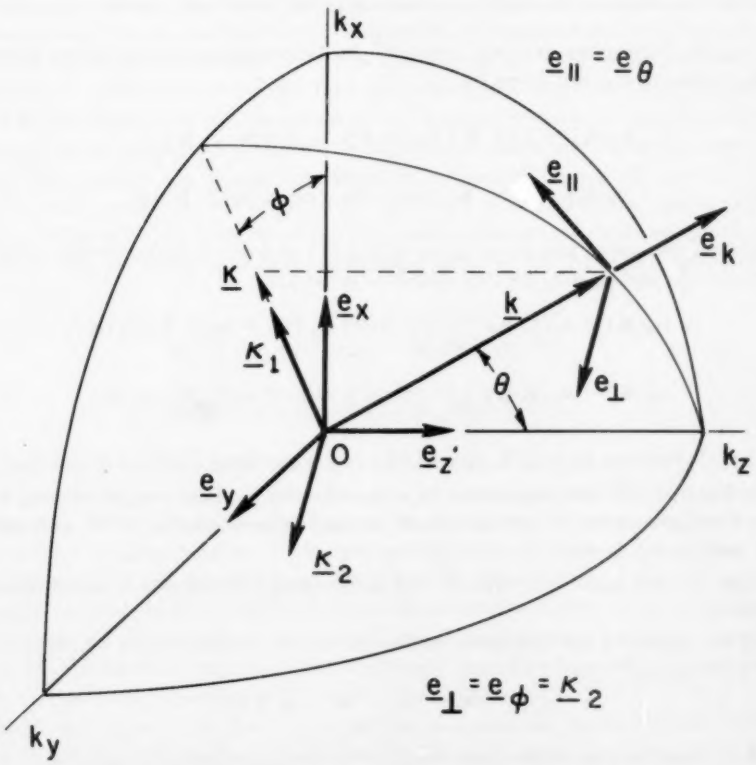

FIGURE 3. *Unit vectors associated with* $\mathbf{k}$ *and* $\mathbf{K}$ *(for real* $\mathbf{k}$*).* See also table 1.

TABLE 1. *Unit Vectors Associated With* **K** *and* **k**

K: $(k_x, k_y) = (K\cos\phi, K\sin\phi)$
$\boldsymbol{\kappa}_1 = \mathbf{K}/K = (k_x\mathbf{e}_x + k_y\mathbf{e}_y)/K = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$
$\boldsymbol{\kappa}_2 \equiv \mathbf{e}_z \times \boldsymbol{\kappa}_1 = (-k_y\mathbf{e}_x + k_x\mathbf{e}_y)/K = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y$
$[\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\mathbf{e}_z] = 1, \qquad \boldsymbol{\kappa}_m(-\mathbf{K}) = -\boldsymbol{\kappa}_m(\mathbf{K})$
k: $(k_x, k_y, k_z) = (k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta)$
$\mathbf{e}_k \equiv \mathbf{k}/k = (K\boldsymbol{\kappa}_1 + k_z\mathbf{e}_z)/k = \sin\theta\,\boldsymbol{\kappa}_1 + \cos\theta\,\mathbf{e}_z$
$\mathbf{e}_\parallel \equiv \boldsymbol{\kappa}_2 \times \mathbf{e}_k = (k_z\boldsymbol{\kappa}_1 - K\mathbf{e}_z)/k = \cos\theta\,\boldsymbol{\kappa}_1 - \sin\theta\,\mathbf{e}_z$
$\mathbf{e}_\perp \equiv \boldsymbol{\kappa}_2$
$[\mathbf{e}_\parallel\mathbf{e}_\perp\mathbf{e}_k] = 1, \qquad \mathbf{e}_\parallel(-\mathbf{k}) = \mathbf{e}_\parallel(\mathbf{k}), \qquad k_z = \pm\gamma$

Notes:

(a) $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ are respectively "radial" and "tangential" unit vectors associated with the plane polar coordinates for **K**. These unit vectors are always real.

(b) In the propagating regime, where $\mathbf{e}_\parallel$ as well as $\mathbf{e}_\perp$ is real, $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ may also be identified as the customary θ and ϕ unit-vectors of the spherical polar coordinates for **k**. The "middle" forms in the table help show what happens when $K > k$: $\mathbf{e}_\parallel$ and $\mathbf{e}_k$ become complex, but remain unit vectors in the sense $\mathbf{e}_\parallel \cdot \mathbf{e}_\parallel = \mathbf{e}_k \cdot \mathbf{e}_k = 1$.

(c) The unit vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$, and $\mathbf{e}_\parallel$ are not defined by the equations in the table at the singular points $K = 0$ and $\theta = 0, \pi$. For $\mathbf{k} = k\mathbf{e}_z$, one may, e.g., determine a consistent set by choosing $\boldsymbol{\kappa}_1 = \mathbf{e}_x$. (cf. use of (4.1–4)).

Here $\mathbf{b}_q$ and $\mathbf{a}_q$, the "complete vectorial spectra" for the outgoing and incoming plane-wave components of $\mathbf{E}_q$, respectively, are given by

$$\mathbf{b}_q(\mathbf{K}) = b_q(1,\mathbf{K})\ (\boldsymbol{\kappa}_1 \mp K\gamma^{-1}\mathbf{e}_z) + b_q(2,\mathbf{K})\ \boldsymbol{\kappa}_2, \tag{1.2–11a}$$

$$\mathbf{a}_q(\mathbf{K}) = a_q(1,\mathbf{K})\ (\boldsymbol{\kappa}_1 \pm K\gamma^{-1}\mathbf{e}_z) + a_q(2,\mathbf{K})\ \boldsymbol{\kappa}_2. \tag{1.2–11b}$$

(The association of the upper and lower signs with $q = 1$ and $q = 2$, respectively, is continued from (7).) Alternatively, $\mathbf{b}_q$ and $\mathbf{a}_q$ may be expressed as follows:

$$\mathbf{b}_q(\mathbf{K}) = b_q(1,\mathbf{K})\ [(-)^{q-1}k/\gamma]\ \mathbf{e}_\parallel(\mathbf{k}) + b_q(2,\mathbf{K})\,\mathbf{e}_\perp(\mathbf{K}), \tag{1.2–12a}$$

$$\mathbf{a}_q(\mathbf{K}) = a_q(1,\mathbf{K})\ [(-)^{q}k/\gamma]\ \mathbf{e}_\parallel(\mathbf{k}) + a_q(2,\mathbf{K})\,\mathbf{e}_\perp(\mathbf{K}), \tag{1.2–12b}$$

where $\mathbf{e}_\parallel(\mathbf{k})$ is defined as $\boldsymbol{\kappa}_2 \times \mathbf{k}/k$, and $\mathbf{e}_\perp(\mathbf{K})$ is a suggestive alternative notation for $\boldsymbol{\kappa}_2$. (The factors in brackets in (12) are introduced as a convenient, explicit way of helping to keep signs straight.) The complete vectorial spectra are of interest to us primarily in the propagating regime, where $\mathbf{e}_\parallel$, as well as $\mathbf{e}_\perp$, is real.

All the unit vectors associated with **k** and **K** are pictured in figure 3 and fully identified and related in table 1.

The complete vectorial spectra must and do satisfy the transversality relations

$$\mathbf{k}^{\pm} \cdot \mathbf{b}_q = 0, \qquad \mathbf{k}^{\mp} \cdot \mathbf{a}_q = 0, \tag{1.2–13}$$

which indeed are expressions of the transversality of the basis fields (5) and (6).

We next observe that (8a) can be written

$$\mathbf{E}_{qt}(\mathbf{r}) = \frac{1}{2\pi} \int [\mathbf{B}_q(\mathbf{K})\, e^{i\gamma|z|} + \mathbf{A}_q(\mathbf{K})\, e^{-i\gamma|z|}]\, e^{i\mathbf{K}\cdot\mathbf{R}} d\mathbf{K}, \tag{1.2-14}$$

where $\mathbf{B}_q$ and $\mathbf{A}_q$, the "transverse vectorial spectra" for the outgoing and incoming plane-wave components of $\mathbf{E}_{qt}$, respectively, are given by

$$\mathbf{B}_q(\mathbf{K}) = \sum_m b_q(m, \mathbf{K})\boldsymbol{\kappa}_m, \tag{1.2-15a}$$

$$\mathbf{A}_q(\mathbf{K}) = \sum_m a_q(m, \mathbf{K})\boldsymbol{\kappa}_m. \tag{1.2-15b}$$

Further we note that, given (15), z-components can be recovered with the aid of the transversality relations (13) or simply by inspection of (11).

Remarks: (a) One may observe a switch—from scalar spectra and vector waves in (7) and (8) to vector spectra and scalar waves in (10) and (14). Equation (7) remains fundamental. (b) The use of the lower case letters **a** and **b** to denote complete spectra and the use of the capitals **A** and **B** to denote the corresponding transverse parts should be noted. This parallels our use of the letters **r, k** and **R, K.**

The representations (7) and (8) afford a similar and essential resolution of any electromagnetic field into two major parts: that represented by incoming waves and that represented by outgoing waves. The role of the former part is identified variously by terms such as incident, exciting, primary, or incoming; the latter, by terms such as induced, secondary, radiated, reradiated, scattered, or outgoing. The scattering matrix, defined in the next section, will be seen to be a way of specifying the linear transformation from the first part of the field to the second.

The asymptotic relation between the far-field values of a scattered or radiated field and its spectrum is of essential interest and importance. This relation reveals the result of the interference among the waves of the continuous spectrum at large distances in any chosen direction of observation. Normally this interference results (remarkably) in the well-known exp $(ikr)/r$ variation with distance. This normal result does not apply to the spectrum of a plane wave, and it might not apply to the radiation or reradiation from a source of infinite size in one or more dimensions. For an antenna of finite size radiating into 3-dimensional space, we do have the asymptotic relations [33, p. 750]

$$\mathbf{E}^r_{qt}(\mathbf{r}) \sim -ik \mid \cos\theta \mid \mathbf{B}_q(\mathbf{R}k/r)\, e^{ikr}/r, \tag{1.2-16a}$$

$$\mathbf{E}^r_{q}(\mathbf{r}) \sim -ik \mid \cos\theta \mid \mathbf{b}_q(\mathbf{R}k/r) e^{ikr}/r. \tag{1.2-16b}$$

Here the superscript "r" refers to the radiated or reradiated component of the field and θ is the polar angle of $\mathbf{r}$ with respect to the z axis. The first of the two equations relates to (14) and the second to (10); they differ only in the presence or absence of the z-component. Both equations are valid whether or not incident waves are also present in the field. Note that we have written $\mathbf{R}k/r$ for $\mathbf{K}$ as the argument of the spectral functions. This expresses the fact that the vectors $\mathbf{r}$ and $\mathbf{k}$ involved must be parallel. In fact if we introduce spherical coordinates for $\mathbf{r}$ such that $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, we see that $\mathbf{b}_q$ and $\mathbf{B}_q$ are expressed as functions of the angular coordinates of $\mathbf{r}$. Furthermore we note that here only real directions of propagation come into consideration, so that γ is real and $\gamma = k \mid \cos\theta \mid$ is valid.

To conclude this subsection, we give a useful, nonphysical, special result, which offers a sharp contrast to the type of results given just above. Namely, if an electromagnetic field is every-

where the (simple or evanescent) plane wave with $\mathbf{E}=\boldsymbol{\alpha}\exp(i\mathbf{k}_{\bar{0}}\cdot\mathbf{r})/2\pi$, then from (9) the corresponding spectral functions are found to be

$$\left.\begin{aligned} a_1(m,\mathbf{K}) &= b_2(m,\mathbf{K}) = \alpha_m\delta(\mathbf{K}-\mathbf{K}_0), \\ b_1(m,\mathbf{K}) &\equiv a_2(m,\mathbf{K}) \equiv 0, \end{aligned}\right\} \tag{1.2-17}$$

where $\alpha_m \equiv \boldsymbol{\kappa}_m\cdot\boldsymbol{\alpha}$ and $\delta(\mathbf{K}-\mathbf{K}_0)$ is an abbreviation for the delta-function product $\delta(k_x-k_{ox})\,\delta(k_y-k_{oy})$. The vector

$$\boldsymbol{A} \equiv \alpha_1\boldsymbol{\kappa}_1+\alpha_2\boldsymbol{\kappa}_2 \tag{1.2-18}$$

will be known as the transverse spectral vector associated with the considered plane wave.

1.3. Definition of Scattering Matrices for Antennas and Scatterers

Let us first consider that a passive material structure, exhibiting linear electromagnetic behavior, is present in the region $z_1 < z < z_2$. The scattering equations are written

$$b_q(m,\mathbf{K}) = \sum_p \int_{\mathrm{L}} \sum_n S_{qp}(m,\mathbf{K};\,n,\mathbf{L})\,a_p(n,\mathbf{L})\,d\mathbf{L}, \qquad (q=1,2) \tag{1.3-1}$$

where, in addition to the summation and integration conventions noted following (1.2-8), we have summation over the index p, giving the contributions from the waves incident both from the "right" and from the "left." The processes described by the functions S_{11} and S_{22} will be called backscattering; those described by S_{12} and S_{21} will be called transcattering. Essentially the same definitions and an example of (1) may be found in [1].

If the scattering object is also an antenna, the scattering matrix must include the transmitting and receiving characteristics, and the scattering equations are written

$$b_0 = S_{00}a_0 + \sum_p \int_{\mathrm{L}} \sum_n S_{0p}(n,\mathbf{L})\,a_p(n,\mathbf{L})\,d\mathbf{L}, \tag{1.3-2a}$$

$$b_q(m,\mathbf{K}) = S_{q0}(m,\mathbf{K})\,a_0 + \sum_p \int_{\mathrm{L}} \sum_n S_{qp}(m,\mathbf{K};n,\mathbf{L})\,a_p(n,\mathbf{L})\,d\mathbf{L}. \tag{1.3-2b}$$

Here $q=1,2$ and we have made use of the quantities b_0 and a_0, defined in (1.1-1). The quantity S_{00} represents "backscattering" observed at S_0 in the feed waveguide, and the functions $S_{0q}(m,\mathbf{K})$ and $S_{q0}(m,\mathbf{K})$ respectively represent the receiving and the transmitting characteristics of the antenna. (The quantities bearing the subscripts q or $p=2$ in (2) represent the desired generalization of the antenna scattering matrix originally defined by Kerns and Dayhoff [1].)

The definition of the antenna scattering matrix is now literally complete. We do not wish to belabor the generality of the definition, but we do call attention to the absence of restrictive assumptions. At best, its full significance can be made apparent only gradually.

It will be advantageous to have the basic scattering equations expressed in vector-dyadic form. In addition to the transverse vectors $\mathbf{B}_q$ and $\mathbf{A}_q$ defined in (1.2-15), we introduce the vectorial transmitting characteristic

$$\mathbf{S}_{q0}(\mathbf{K}) = \sum_m S_{q0}(m,\mathbf{K})\,\boldsymbol{\kappa}_m, \tag{1.3-3a}$$

the vectorial receiving characteristic

$$\mathbf{S}_{0q}(\mathbf{K}) = \sum_m S_{0q}(m,\mathbf{K})\ \boldsymbol{\kappa}_m, \tag{1.3-3b}$$

and the dyadic scattering characteristic

$$\mathbf{S}_{qp}(\mathbf{K},\mathbf{L}) = \sum_{m,n} S_{qp}(m,\mathbf{K};n,\mathbf{L})\ \boldsymbol{\kappa}_m \boldsymbol{\lambda}_n, \tag{1.3-3c}$$

where the $\boldsymbol{\kappa}_m$ and $\boldsymbol{\lambda}_n$ are the unit vectors associated respectively with $\mathbf{K}$ and $\mathbf{L}$. The scattering equations become

$$b_0 = S_{00}a_0 + \sum_p \int \mathbf{S}_{0p}(\mathbf{K})\cdot\mathbf{A}_p(\mathbf{K})\,d\mathbf{K} \tag{1.3-4a}$$

$$\mathbf{B}_q(\mathbf{K}) = \mathbf{S}_{q0}(\mathbf{K})a_0 + \sum_p \int \mathbf{S}_{qp}(\mathbf{K},\mathbf{L})\cdot\mathbf{A}_p(\mathbf{L})\,d\mathbf{L}. \tag{1.3-4b}$$

The equations are now invariant with respect to choice of coordinates in the transverse plane. This facilitates discussion of "real world" cases, in which it is neither expedient nor necessary always to observe or produce pure $\boldsymbol{\kappa}_1$ or $\boldsymbol{\kappa}_2$ field components.

One may obtain an analysis of the scattering equations by considering the simplest modes of excitation: by a wave represented by a_0 alone, and by individual spatial plane waves, represented by delta-function spectra. This procedure leads essentially to re-expressions of the basic definitions contained in the scattering equations. The circumstances of the resulting definitions or expressions are simple enough to suggest several more or less direct and conventional methods of measurement for the scattering-matrix elements. Let us in particular consider the definitions of $\mathbf{S}_{q0}(\mathbf{K})$ and $\mathbf{S}_{0q}(\mathbf{K})$—the scattering-matrix quantities that will receive the most attention in our work.

Let us consider an antenna operating in its transmitting mode; that is, an antenna excited only by an incident wave in its waveguide feed and radiating into empty space. This elementary pattern of excitation is represented by $a_0 \neq 0$, $a_q(m,\mathbf{K}) \equiv 0$. Under these conditions, the spectra radiated (to the left and to the right), normalized to unit a_0, characterize the transmitting properties of the antenna; indeed, from (4b)

$$\mathbf{S}_{q0}(\mathbf{K}) = \frac{\mathbf{B}_q(\mathbf{K})}{a_0}. \tag{1.3-5}$$

Incident spatial waves being absent, (1.2–9a) and (1.2–9b) together imply that $\mathbf{B}_q$ may be related *either* to $\mathbf{E}$ or to $\mathbf{H}$; in terms of $\mathbf{E}$ we have

$$\mathbf{S}_{q0}(\mathbf{K}) = \frac{e^{-i\gamma|z|}}{2\pi a_0}\int \mathbf{E}_{qt}(\mathbf{R},z)e^{-i\mathbf{K}\cdot\mathbf{R}}\,d\mathbf{R}, \tag{1.3-6}$$

where $z > z_1$ or $z < z_2$ must be in force. This gives us a Fourier transform definition of $\mathbf{S}_{q0}$, and, to the extent that $\mathbf{E}_{qt}/a_0$ is known or measurable, a means of calculating $\mathbf{S}_{q0}$. The asymptotic relation (1.2–16a), applied in the present circumstances, may be written

$$\mathbf{E}_{qt}(\mathbf{r}) \sim -i\gamma\mathbf{S}_{q0}(\mathbf{R}k/r)a_0 e^{ikr}/r. \tag{1.3-7}$$

Clearly this relation may be regarded as a formula for determining $\mathbf{S}_{q0}(\mathbf{K})$ in terms of observed asymptotic $\mathbf{E}_{qt}/a_0$. It in fact represents the basis of so-called direct methods of measurement of transmitting characteristics.

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The combination of (6) and (7) incidentally furnishes a rigorous, vectorial form of the Fourier-transform relation between far- and near-fields.

Next, consider an antenna excited solely by a spatial plane wave, incident on side q, say, and having $\mathbf{E} = \boldsymbol{\alpha} \exp(i\mathbf{k}\cdot\mathbf{r})/2\pi$. The pattern of excitation is accordingly represented by $a_0 = 0$ and $a_{q'}(m,\mathbf{K}') = \alpha_m \delta_{q'q} \delta(\mathbf{K}' - \mathbf{K})$, where $\alpha_m \equiv \boldsymbol{\kappa}_m \cdot \boldsymbol{\alpha}$ and the spectrum of the incident wave is found just as in (1.2–17). The scattering equations (2a) or (4a) now yield

$$\begin{aligned} b_0 &= S_{0q}(1, \mathbf{K})\alpha_1 + S_{0q}(2, \mathbf{K})\alpha_2, \\ &= \mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}. \end{aligned} \tag{1.3–8}$$

In other words, $S_{0q}(m,\mathbf{K})$ denotes the receiving sensitivity—or receptivity—of the antenna to the $\boldsymbol{\kappa}_m$-component of polarization of a plane wave incident on the antenna with direction of incidence specified by q and $\mathbf{K}$. The normalization is to unit α_m (which means unit delta-function spectrum amplitude).

Equation (8) immediately suggests basic equations for direct measurement of receiving characteristics. Viz.,

$$\begin{aligned} b_0' &= \mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}', \\ b'' &= \mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}'', \end{aligned} \tag{1.3–9}$$

where b_0', b_0'' are observed and $\boldsymbol{A}', \boldsymbol{A}''$ are known and linearly independent. The equation can, of course, be solved for $\mathbf{S}_{0q}$; an algebraically identical problem is encountered in subsection 4.1.

The following compact notation for the scattering equations, to be secured by introducing function vectors and making more use of matrix-algebraic concepts, provides a perspective quite different from that of the discussion just preceding. We first define the column matrices

$$\begin{pmatrix} a_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \qquad \begin{pmatrix} b_0 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}, \tag{1.3–10}$$

in which $\hat{a}_q$ and $\hat{b}_q$ may themselves be regarded as column matrices representing the functions $a_q(m,\mathbf{K})$ and $b_q(m,\mathbf{K})$. That is to say, the elements of these column matrices are labeled or indexed according to the values of m and $\mathbf{K}$, and have the values $a_q(m,\mathbf{K})$ and $b_q(m,\mathbf{K})$, respectively. The transformation from the entire set of incident waves to the entire set of emergent waves is now written

$$\begin{pmatrix} b_0 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} S_{00} & \hat{S}_{01} & \hat{S}_{02} \\ \hat{S}_{10} & \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{20} & \hat{S}_{21} & \hat{S}_{22} \end{pmatrix} \begin{pmatrix} a_0 \\ \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{1.3–11}$$

or, equivalently, after performing the indicated matrix multiplication,

$$\begin{aligned} b_0 &= S_{00}a_0 + \hat{S}_{01}\hat{a}_1 + \hat{S}_{02}\hat{a}_2, \\ \hat{b}_1 &= \hat{S}_{10}a_0 + \hat{S}_{11}\hat{a}_1 + \hat{S}_{12}\hat{a}_2, \\ \hat{b}_2 &= \hat{S}_{20}a_0 + \hat{S}_{21}\hat{a}_1 + \hat{S}_{22}\hat{a}_2. \end{aligned} \tag{1.3–12}$$

Here the four kinds of products involved are defined by comparison with (2) or (4). This compact notation, in which the subscripts identify the three input-output reference surfaces, makes the overall structure of the equations more apparent and is a practical necessity for the demonstration in the next section. The rules of matrix algebra apply: the $\hat{S}_{q0}$ (as well as $\hat{a}_q$ and $\hat{b}_q$) correspond to column matrices; the $\hat{S}_{0q}$, to row matrices; and the $\hat{S}_{pq}$, to square matrices.

We should notice the form that the scattering equations take in the absence of any scattering object (or conceivably in the presence of a non-scattering object). The free passage of waves is expressed by

$$b_1(m,\mathbf{K}) \equiv a_2(m,\mathbf{K}), \qquad b_2(m,\mathbf{K}) \equiv a_1(m,\mathbf{K}). \tag{1.3-13}$$

(No propagation factors of the form $\exp(\pm i\gamma d)$ appear here because the phase reference surfaces were chosen coincident at $z = 0$). The pertinent submatrix in (11) is

$$\begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix} = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix} \tag{1.3-14}$$

Here $\hat{1}$ denotes the identity transformation with elements $\delta_{mn}\delta(k_x - l_x)\,\delta(k_y - l_y)$ and $\hat{0}$ denotes the zero transformation. One may regard a scattering object as producing a perturbation of the properties of free-space described in (13) or (14). From this viewpoint, what we may call true or *bona fide* transcattering is described by the operators $\hat{S}_{12} - \hat{1}$ and $\hat{S}_{21} - \hat{1}$.

1.4. Power Expressions

The requisite expression for one-mode power transfer in the antenna feed waveguide was given in (1.1–4). Here we shall give the corresponding expressions for power transfer across the surfaces F_1 and F_2.

The time-average energy flux across the surface F_q in the outward direction (the direction of $\mathbf{n}_q$) is given by

$$P_q = \frac{1}{2}\,\mathrm{Re}\int_{F_q} [\mathbf{E}\bar{\mathbf{H}}\mathbf{n}_q]\,d\mathbf{R}, \tag{1.4-1}$$

where the integrand is the scalar triple product and the superposed bar denotes the complex conjugate. We wish to evaluate P_q in terms of the spectral density functions $a_q(m,\mathbf{K})$ and $b_q(m,\mathbf{K})$. After some analysis, one obtains

$$P_q = \frac{1}{2}\int_{K<k}\sum_m (|b_q|^2 - |a_q|^2)\eta_m\,d\mathbf{K} - \int_{K>k}\sum_m \mathrm{Re}\,(\bar{b}_q a_q \eta_m)\,d\mathbf{K}, \tag{1.4-2}$$

noting that for the lossless medium γ and η_m are both real in the propagating region ($K < k$) and both imaginary in the evanescent region ($K > k$). The equation shows that power may be transferred by the coupling of incoming and outgoing evanescent modes having the same m and $\mathbf{K}$. (This the essential mechanism of power transfer in a waveguide-below-cutoff attenuator.) We expect and assume that ordinarily this interaction will be negligible, as in the case of coupling of ordinary waveguide junctions; the separation required is measured in wavelengths, not Rayleigh distances. Thus the basic expression (2) may ordinarily be abbreviated to

$$P_q = \frac{1}{2}\int_{K<k}\sum_m [|b_q(m,\mathbf{K})|^2 - |a_q(m,\mathbf{K})|^2]\eta_m(K)\,d\mathbf{K}. \tag{1.4-3}$$

This may be used, e.g., to deduce conservation relations. In the absence of incident waves, (3) expresses the equivalent of "pattern integration" over a hemisphere.

1.5. Reciprocal Relations—A Summary

In this section we briefly state recently derived [34] generalized or *adjoint reciprocity* relations for antennas and scatterers. (The derivation is reproduced in appendix A.) These relations very readily adapt to the expression of ordinary reciprocity as a special case. We shall also comment on the question of "realizability" of mutually adjoint systems. First we must define mutually adjoint media and systems.

We describe the distribution of material media making up an antenna or scattering structure by means of the constitutive equations

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E} + \boldsymbol{\tau} \cdot \mathbf{H}, \qquad \mathbf{B} = \boldsymbol{\nu} \cdot \mathbf{E} + \boldsymbol{\mu} \cdot \mathbf{H}. \tag{1.5-1}$$

Here the tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ have their usual roles; $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ allow for the description of possible magnetoelectric properties of the medium [35].[5] (This last bit of generality may provide future benefits and does not appreciably complicate the discussion.) The tensor parameters will of course in general depend upon position within the region of the antenna or scatterer considered; outside this region the set of parameters must reduce nominally to vacuum values: $\boldsymbol{\epsilon} = \epsilon_0$, $\boldsymbol{\mu} = \mu_0$, $\boldsymbol{\tau} = \boldsymbol{\nu} = 0$.

In addition to a "given" or "original" system, described by the above equations, we must consider the *adjoint* system, which is described by the constitutive equations for the adjoint medium,[6]

$$\mathbf{D} = \tilde{\boldsymbol{\epsilon}} \cdot \mathbf{E} - \tilde{\boldsymbol{\nu}} \cdot \mathbf{H}, \quad \mathbf{B} = -\tilde{\boldsymbol{\tau}} \cdot \mathbf{E} + \tilde{\boldsymbol{\mu}} \cdot \mathbf{H}, \tag{1.5-2}$$

where the superposed tilde denotes the transpose, and the tensors $\boldsymbol{\epsilon}$, $\boldsymbol{\tau}$, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$ are those of the original system. As the equations show, "adjointness" is a mutual relationship: the adjoint of the adjoint system is the original system.

Since the tensor parameters are essentially arbitrary, we should realize that the "original" system is in no way a theoretically preferred system; the designation is arbitrary but useful.

The concept that a medium may be lossy, lossless, or even "gainy" is familiar. In an inhomogeneous medium these "dissipative properties" will in general change from point to point. In appendix B dissipative properties are more precisely defined and it is shown that these properties are point-wise identical for mutually adjoint media. In view of the usual connection between dissipative properties and the concept of "realizability," we may say that mutually adjoint media are equally realizable.

Nonreciprocal antennas are most commonly (if not invariably) so because of the use of ferrites subjected to a static magnetic biasing field. In such cases the adjoint antenna can in principle be produced by reversal of the bias field. (We say "in principle" because in general no provision is made for conveniently or precisely accomplishing the bias field reversal.)

The scattering matrix for the adjoint antenna may be and is assumed to be defined with the same basis fields, the same reference surfaces, and altogether in the same way as that for the original antenna. Then, as shown in appendix A, the following generalized or *adjoint reciprocity relations* hold between the characteristics of mutually adjoint antennas: For the antenna-feed reflection coefficients,

$$S_{00} = S_{00}^{a}; \tag{1.5-3}$$

[5] In some of the recent literature, media described by equations of the form of (1) are called "bianisotropic." However, optically active media, which have been studied for many years [36], satisfy equations of the above form with scalar parameters and are not anisotropic at all. Optically active media incidentally are reciprocal, and thus could form part of a reciprocal antenna or other device.

[6] Media related by (1) and (2) are called "complementary" media by Kong and Cheng [37]. The adjective "adjoint" seems more appropriate both nontechnically and technically: Maxwell's equations for the adjoint system can be written as the mathematical adjoint of Maxwell's equations for the original system. See appendix A.

for the transmitting and the receiving characteristics,

$$\eta_0 S^a_{0q}(m, \mathbf{K}) = -\eta_m(K) S_{q0}(m, -\mathbf{K}), \qquad (q=1,2) \tag{1.5-4a}$$

$$\eta_0 S_{0q}(m, \mathbf{K}) = -\eta_m(K) S^a_{q0}(m, -\mathbf{K}); \quad (q=1,2) \tag{1.5-4b}$$

and for the scattering characteristics,

$$\eta_m(K) S_{pq}(m, \mathbf{K}; n, \mathbf{L}) = \eta_n(L) S^a_{pq}(n, -\mathbf{L}; m, -\mathbf{K}). \qquad (p=1,2; q=1,2) \tag{1.5-5}$$

The superscript "a" distinguishes quantities associated with the adjoint antenna. We observe that all the equations hold with S and S^a interchanged.

Scattering-matrix elements, such as $S_{10}(m, \mathbf{K})$ and $S_{11}(m, \mathbf{K}, n, \mathbf{L})$, characterize processes associated with a pair of directions:[7] the direction of an incident wave and a direction of "observation." For the functions mentioned, the direction-pairs are $\mathbf{n}_0$, $\mathbf{e}_k$ and $\mathbf{e}_l$, $\mathbf{e}_k$, respectively. Reciprocity relates processes associated with two pairs of directions, obtained by reversing and interchanging the direction of incidence and the direction of observation. See figures 4 and 5.

If the constitutive tensors obey the symmetry relations

$$\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}}, \ \boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}, \text{ and } \boldsymbol{\tau} = -\tilde{\boldsymbol{\nu}}, \tag{1.5-6}$$

then, as may be seen directly from (1) and (2), the adjoint antenna and the original antenna are identical. In this case, if we use conventional terminology, we say that the original antenna is reciprocal; in the present context, a term such as self-reciprocal or self-adjoint would be less liable to ambiguity. In the self-adjoint case the superscript "a" is without force and may be eliminated; (3), (4), and (5) become expressions of properties of one and the same antenna (and reduce to results given with $p = q = 1$ in [1]).

Equations (3), (4), and (5) are our basic expressions of reciprocity and adjoint reciprocity. As mentioned in the Introduction, these relations have found substantial application in research establishing the foundations of the extrapolation technique [6,7]. More immediate consequences are brought out in the next subsection and in section 4.

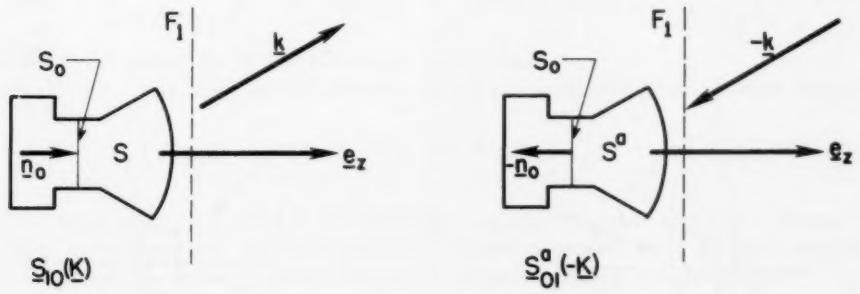

FIGURE 4. *Reciprocity direction-diagram for* $\mathbf{S}_{10}$ *and* $\mathbf{S}^a_{01}$.

[7] This is in marked contrast to what is involved when spherical or other nonplanar waves are used to represent the fields in space.

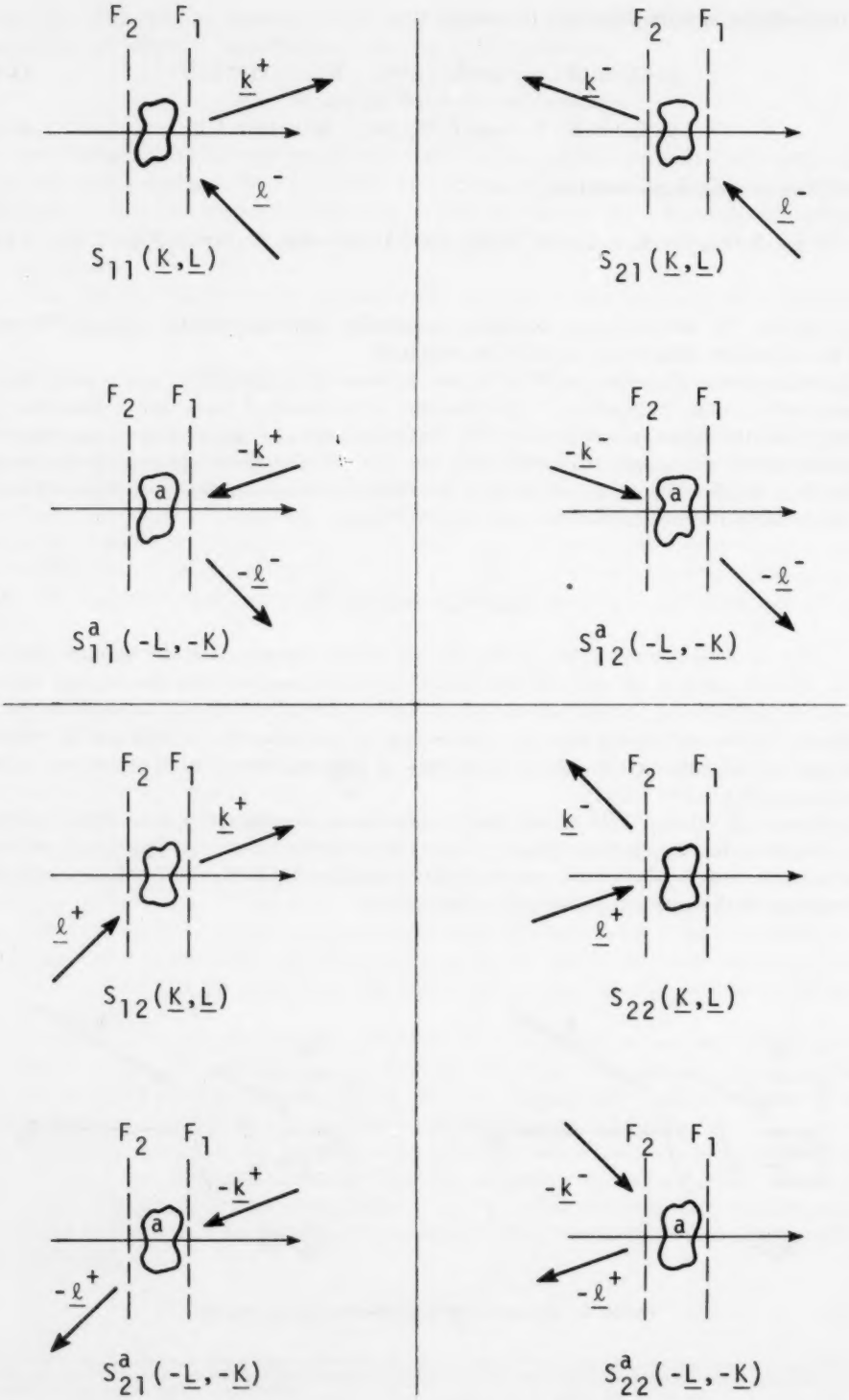

FIGURE 5. *Direction-diagrams for the scattering reciprocity relations.*

1.6. Power Gain, Effective Area, and Polarization Indices

Our primary purpose in this subsection is to exhibit some of the content and applicability of the PWSM formalism by defining some of the more familiar and conventional quantities of antenna theory in terms of the antenna scattering matrix quantities. In particular, we shall define the power-gain function G_q and the effective-area function σ_q associated respectively with the transmitting and receiving characteristics $\mathbf{S}_{q0}$ and $\mathbf{S}_{0q}$. We shall also define polarization indices ρ_{q0} and ρ_{0q} associated respectively with $\mathbf{S}_{q0}$ and $\mathbf{S}_{0q}$. All the quantities to be defined are of course functions of direction and all are meaningful and applicable in dealing with the coupling of widely separated antennas, the context of conventional antenna theory. It will be seen, however, that although these quantities are uniquely defined (in a given coordinate system) by the antenna scattering-matrix quantities, the converse is not true. Hence they are not adequate characterizations of transmitting and receiving properties for the type of theory of essential interest in this paper (and, of course, they say nothing about scattering properties of an antenna).

Our definitions of power gain and effective area are consistent with the essential content of the corresponding IEEE Standard definitions. However, our definitions of receiving and of transmitting properties are formed wholly independently of each other—in contrast to a tactic used to some extent in the IEEE Standards. Reciprocity relations then appear only later in their proper role as theorems.

The key relation appropriate for the present consideration of transmitting characteristics is the asymptotic relation

$$\mathbf{E}_q(\mathbf{r}) \sim -i\gamma \mathbf{s}_{q0}(\mathbf{R}k/r) a_0 e^{ikr}/r. \tag{1.6-1}$$

Here

$$\mathbf{s}_{q0}(\mathbf{K}) a_0 = \mathbf{b}_q(\mathbf{K}), \tag{1.6-2}$$

where $\mathbf{b}_q$ is the complete radiated spectrum in the sense of (1.2–12a) and $\mathbf{s}_{q0}$ is the corresponding "complete" transmitting characteristic given in terms of components of $\mathbf{S}_{0q}$ by

$$\mathbf{s}_{q0}(\mathbf{K}) = [(-)^{q-1} k/\gamma] S_{q0}(1,\mathbf{K})\, \mathbf{e}_{\parallel} + S_{q0}(2,\mathbf{K})\, \mathbf{e}_{\perp}. \tag{1.6-3}$$

The following relations are noted:

$$\mathbf{k}^{\pm} \cdot \mathbf{s}_{q0}(\mathbf{K}) = 0, \qquad \mathbf{s}_{q0}(0) = \mathbf{S}_{q0}(0),$$

and $\mathbf{S}_{q0}(\mathbf{K})$ is the projection of $\mathbf{s}_{q0}(\mathbf{K})$ on the k_x, k_y plane.

From (1) one easily finds for the power radiated per unit solid angle at large distances

$$p_q = \frac{1}{2} Y_0 \gamma^2 |\mathbf{s}_{q0}(\mathbf{K}) a_0|^2. \tag{1.6-4}$$

The Y_0 appearing here is the value of the admittance $(\epsilon_0/\mu_0)^{1/2}$ for the ambient medium. The function p_q is a "power pattern" for the considered antenna; generally any function proportional to p_q, whether or not the factor of proportionality is known, is called a power pattern.

The *power gain* (function) of an antenna is defined by

$$G_q(\mathbf{K}) = 4\pi p_q / P_0, \tag{1.6-5}$$

where P_0 is the *net input* power to the antenna. It follows from (1.1–4) and (5) that

$$G_q(\mathbf{K}) = \frac{4\pi Y_0 \gamma^2 |s_{q0}(\mathbf{K})|^2}{\eta_0 (1 - |S_{00}|^2)}. \tag{1.6–6}$$

It should be noted that the power gain is a characteristic of the antenna under consideration, independent of the source used to excite the antenna. This means that the value of $G_q(\mathbf{K})$ is independent of the insertion or adjustment of a lossless tuner in the feed waveguide, whether or not this tuner is counted as part of the antenna (cf. discussion of eq (2.6–13) in [30]).

Using the components shown in (3), we define the polarization index

$$\rho_{q0}(\mathbf{K}) = \frac{S_{q0}(2, \mathbf{K})}{S_{q0}(1, \mathbf{K})(-)^{q-1} k/\gamma} \tag{1.6–7}$$

associated with the transmitting characteristics of an antenna. This spectral polarization index is definable more physically and more conventionally in terms of the components of the corresponding asymptotic **E**; by (1), it is just the ratio of the $\perp$ and $\|$ (or ϕ and θ) components of this **E** in the direction of observation. Polarization characteristics are conveniently and fully described by the single complex number ρ_{q0}; other polarization parameters, such as axial ratio and orientation of the associated polarization ellipse, can be determined, if desired, from ρ_{q0}.

For an antenna functioning in a receiving mode, the counterpart of the power gain is the effective area or effective receiving cross-section, $\sigma_q(\mathbf{K})$. Like the gain, this quantity is a scalar function of direction and involves a far-field concept—in this case that of an incident plane wave. It is here defined by

$$P_{A,\max} = \sigma_q(\mathbf{K}) S_{Poy}, \tag{1.6–8}$$

where $P_{A,\max}$ is the available power at the antenna terminal and S_{Poy} is the magnitude of the Poynting's vector of a plane wave arriving at the antenna from a given direction and providing a polarization match to the antenna receiving characteristic. The derivation of an expression for σ_q involves a number of intermediate results that are at least as important as σ_q itself.

Let the electric field of the wave incident on side q of the antenna be $\mathbf{E} = \boldsymbol{\alpha} \exp(i\mathbf{k}\ \mathbf{r})/2\pi$; then from (1.3–2a) or (1.3–4a), the emergent wave-amplitude at the antenna terminal is

$$b_0 = S_{00} a_0 + \mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}. \tag{1.6–9}$$

(This expression differs from (1.3–8) only in that here we have not assumed $a_0 = 0$.) Thus, from (9), we see that the antenna, as excited by the incident spatial wave, appears at S_0 as a source having a reflection coefficient S_{00} and a generated wave $b_G = S_{0q}(\mathbf{K}) \cdot \boldsymbol{A}$. By ordinary (microwave) circuit calculation the corresponding received power is found to be

$$P_{\text{rec}} = \frac{1}{2} \eta_0 \frac{(1 - |\Gamma_L|^2) |\mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}|^2}{|1 - \Gamma_L S_{00}|^2}, \tag{1.6–10}$$

where Γ_L is the reflection coefficient of the passive termination at S_0. By setting $\Gamma_L = \bar{S}_{00}$, we get for the available power

$$P_A = \frac{1}{2} \eta_0 \frac{|\mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A}|^2}{1 - |S_{00}|^2}. \tag{1.6–11}$$

To aid in the consideration of the polarization-related parts of the problem, we display the spectral vector

$$\boldsymbol{A} = \alpha_1 \boldsymbol{\kappa}_1 + \alpha_2 \boldsymbol{\kappa}_2$$

used here and originally defined in (1.2–18); we introduce the complete spectral vector for the incident wave in the form

$$\boldsymbol{\alpha}=[(-)^q k/\gamma]\alpha_1 \mathbf{e}_{||}+\alpha_2 \mathbf{e}_{\perp}; \tag{1.6–12}$$

and we define the polarization index

$$w_q(\mathbf{K}) \equiv \frac{\alpha_2}{\alpha_1(-)^q k/\gamma} \tag{1.6–13}$$

for the incident wave.

Further, it is convenient to define a receiving characteristic, $\mathbf{s}_{0q}$, complementary to the complete transmitting characteristic $\mathbf{s}_{q0}$. The single essential requirement is that

$$\mathbf{s}_{0q}(\mathbf{K}) \cdot \boldsymbol{\alpha}=\mathbf{S}_{0q}(\mathbf{K}) \cdot \boldsymbol{A} \tag{1.6–14a}$$

be an identity in $\boldsymbol{A}$. This leaves a possible $\mathbf{e}_k$-component of $\mathbf{s}_{0q}$ undefined; we are in fact free to require

$$\mathbf{k}^{\mp} \cdot \mathbf{s}_{0q}(\mathbf{K}) \equiv 0. \tag{1.6–14b}$$

(Here, as before, the upper sign goes with $q=1$, the lower, with $q=2$.) Equations (14) and (12) imply

$$\mathbf{s}_{0q}(\mathbf{K})=[(-)^q \gamma/k]S_{0q}(1,\mathbf{K})\mathbf{e}_{||}+S_{0q}(2,\mathbf{K})\mathbf{e}_{\perp}. \tag{1.6–15}$$

One should note particularly that γ/k appears here whereas k/γ appears in the expression for $\mathbf{s}_{q0}$; $\mathbf{s}_{0q}$ is in no way a "complete" vector of which $\mathbf{S}_{0q}$ is a part. However, the relation

$$\mathbf{s}_{0q}(0)=\mathbf{S}_{0q}(0)$$

does hold.

Using the components shown in (15), we define a polarization index for the receiving characteristics

$$\rho_{0q}(\mathbf{K}) \equiv \frac{S_{0q}(2,\mathbf{K})}{S_{0q}(1,\mathbf{K})(-)^q \gamma/k}. \tag{1.6–16}$$

This parameter relates to the properties of a passive material structure; it does not directly characterize the elliptical path of a time-varying vector. It is a ratio of receptivities to components of polarization in an incident plane wave under the specified conditions.

Observing that $|\boldsymbol{\alpha}|^2=8\pi^2 S_{P0y}/Y_0$, we can now combine (11), (14a), (13), and (16) to obtain

$$P_A=\left[\frac{|1+w_q\rho_{0q}|^2}{(1+|w_q|^2)(1+|\rho_{0q}|^2)}\right]\frac{4\pi^2\eta_0|\mathbf{s}_{0q}|^2 S_{P0y}}{Y_0(1-|S_{00}|^2)}. \tag{1.6–17}$$

The quantity in brackets is a "polarization mismatch" factor, which, according to the Schwarz inequality for complex vectors, attains its maximum value of unity for

$$w_q(\mathbf{K})=\overline{\rho_{0q}(\mathbf{K})}. \tag{1.6–18}$$

Thus the condition for polarization match is expressed as a conjugate match of polarization indices. When (18) holds, we find from (17) and (8)

$$\sigma_q(\mathbf{K})=\frac{4\pi^2\eta_0|\mathbf{s}_{0q}(\mathbf{K})|^2}{Y_0(1-|S_{00}|^2)} \tag{1.6–19}$$

as the desired expression for the effective area.

The *minimum* value of P_A, incidentally, is zero: for the given antenna and for any given direction of the incident wave, there is always a wave-polarization, $w_q = -1/\rho_{0q}$, that is not received at all. The polarizations best received and not received are mutually orthogonal (in the power or Hermitian sense).

The complete transmitting characteristic $\mathbf{s}_{q0}$ and the complementary receiving characteristic $\mathbf{s}_{0q}$ for a reciprocal antenna satisfy the reciprocity relation

$$\eta_0 k \mathbf{s}_{0q}(\mathbf{K}) = Y_0 \gamma \mathbf{s}_{q0}(-\mathbf{K}), \tag{1.6-20a}$$

which follows from the basic relations (1.5–4). (In applying (1.5–4) it may be convenient to use the forms $\eta_1 = Y_0 k/\gamma$ and $\eta_2 = Y_0 \gamma/k$ for the η's, which were originally defined in (1.2–5, –6).) For mutually adjoint antennas, the relations corresponding to (20a) are

$$\eta_0 k \mathbf{s}^a_{0q}(\mathbf{K}) = Y_0 \gamma \mathbf{s}_{q0}(-\mathbf{K}), \qquad \eta_0 k \mathbf{s}_{0q}(\mathbf{K}) = Y_0 \gamma \mathbf{s}^a_{q0}(-\mathbf{K}). \tag{1.6-20b}$$

The power-gain and effective-area functions for a reciprocal antenna satisfy the well-known reciprocity relation

$$\sigma_q(\mathbf{K}) = \frac{\lambda^2}{4\pi} G_q(-\mathbf{K}). \tag{1.6-21a}$$

For mutually adjoint antennas the corresponding relations are

$$\sigma^a_q(\mathbf{K}) = \frac{\lambda^2}{4\pi} G_q(-\mathbf{K}), \qquad \sigma_q(\mathbf{K}) = \frac{\lambda^2}{4\pi} G^a_q(-\mathbf{K}). \tag{1.6-21b}$$

The results in this set are conveniently found as corollaries of (20). Relations of the type (21b) were noted by Harrington and Villeneuve for antennas containing gyrotropic media [38].

The reciprocity constraint for the transmitting and receiving polarization characteristics of a reciprocal antenna reads

$$\rho_{0q}(-\mathbf{K}) = -\rho_{q0}(\mathbf{K}). \tag{1.6-22a}$$

The corresponding relations for mutually adjoint antennas are

$$\rho^a_{0q}(-\mathbf{K}) = -\rho_{q0}(\mathbf{K}), \qquad \rho_{0q}(-\mathbf{K}) = -\rho^a_{q0}(\mathbf{K}), \tag{1.6-22b}$$

which follow as further consequences of (1.5–4) or (20).

An interesting corollary of the polarization matching and reciprocity theorems is that if the radiation from a reciprocal antenna in a certain direction is circularly polarized, the wave best received from that direction is circularly polarized in the same screw sense. A verification of this, though basically simple, involves a few key elements. The polarization-index definitions (7) and (13) are so fashioned that $\rho_{q0}(\mathbf{K}) = i$ and $w_q(\mathbf{K}) = i$ both represent right-handed circular polarizations[8] (for $q = 1, 2$ and for all $K < k$). Here the behavior, and in particular the parity, of the unit vectors $\mathbf{e}_{||}$ and $\mathbf{e}_{\perp}$ as functions of $\mathbf{k}$ is directly involved (refer to table 1 and especially to fig. 6). Note that $\mathbf{e}_{||}$, $\mathbf{e}_{\perp}$, and $\mathbf{e}_k$ form a right-handed system congruent to $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. The stated corollary follows upon setting $\rho_{q0}(\mathbf{K}) = \pm i$ and applying (22a) and (18).

[8] See, e.g., Beckmann [39] or Hollis *et al.* [40]. For optical terminology (which differs) see, e.g., Born and Wolf [33, p. 27].

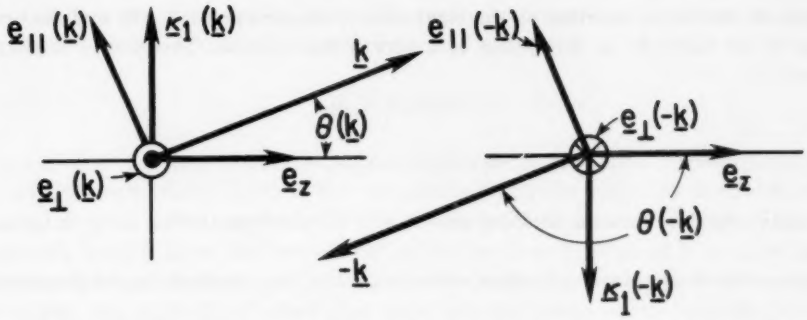

FIGURE 6. *Unit vectors in and perpendicular to the plane of* $\mathbf{e}_z$ *and* $\mathbf{k}$ *for* $\pm\mathbf{k}$, $\mathbf{k}$ *real.*

2. Scattering-Matrix Analysis of Coupled Antennas; General Solution for System 2-Port

We consider a system consisting of a pair of antenna systems operating in a homogeneous, isotropic, dissipationless medium, as shown in the highly schematic figure 7. We are primarily interested in this system as a *transmission system*, with one antenna transmitting and the other receiving. The complete treatment of a transmission system must include effects of scattering by both antennas, and thus automatically includes treatment of *reflection systems*, in which one antenna functions in both transmitting and receiving modes and the other antenna represents an arbitrary passive (linear) scattering object.

For the description of the antenna on the left in figure 7, we apply (1.3–12). In the problem of interest there are no waves incident from the left ($\hat{a}_2=0$); the spectrum of waves going to the left, $\hat{b}_2$, is not involved in the process of solving the problem (but it is obtainable as a part of the solution). Thus (1.3–12) reduces essentially to

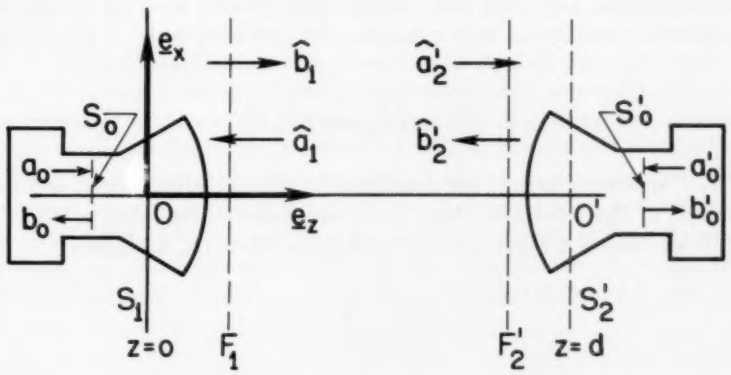

FIGURE 7. *Transmission system—schematic.*

$$b_0 = S_{00}a_0 + \hat{S}_{01}\hat{a}_1,$$
$$\hat{b}_1 = \hat{S}_{10}a_0 + \hat{S}_{11}\hat{a}_1. \tag{2-1}$$

These equations are set up with reference to terminal surfaces S_0 and S_1, the latter being at $z=0$ in the coordinate system $0xyz$. For the description of the antenna on the right, we again apply (1.3–12), using primes to distinguish quantities associated with this antenna. In the problem of

interest there are no waves incident on the right side of this antenna ($\hat{a}_1' = 0$) and the spectrum of waves going to the right, $\hat{b}_1'$, is obtainable as a part of the solution. The needed scattering equations thus are

$$b_0' = S_{00}' a_0' + \hat{S}_{02}' \hat{a}_2', \qquad \hat{b}_2' = \hat{S}_{20}' a_0' + \hat{S}_{22}' \hat{a}_2'. \tag{2-2}$$

For these equations the terminal surfaces are S_0' and S_2', the latter being $z = d$ in the coordinate system $0xyz$.

The separation of the phase reference surfaces for the two antennas by the distance d implies

$$a_2'(m, \mathbf{K}) = b_1(m, \mathbf{K}) e^{i\gamma d}, \qquad a_1(m, \mathbf{K}) = b_2'(m, \mathbf{K}) e^{i\gamma d}. \tag{2-3}$$

Now, with respect to the transmission path as an element of the system, the set of incident waves is represented by $\hat{b}_1$ and $\hat{b}_2'$ and the set of emergent waves by $\hat{a}_1$ and $\hat{a}_2'$. Thus from (3), the matrix description of this element is

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \hat{0} & \hat{T} \\ \hat{T} & \hat{0} \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2' \end{pmatrix} \tag{2-4}$$

where the elements of $\hat{T}$ are $T(\mathbf{K}; \mathbf{L}) = \delta(k_x - l_x)\delta(k_y - l_y) e^{i\gamma(K)d}$. This is equivalent to the two separate transformations

$$\hat{a}_1 = \hat{T}\hat{b}_2', \quad \hat{a}_2' = \hat{T}\hat{b}_1. \tag{2-5}$$

We are now in a position to obtain a complete formal solution for the behavior of the transmission system under consideration. That is, we can obtain expressions for both b_0 and b_0', valid at arbitrary distances and including the effects of multiple reflections. (We can also formally determine the field in the transmission path.) We first consider transmission from left to right, assuming that the receiving antenna is terminated with a passive, reflectionless load. Using (2) and (5), we find

$$\hat{a}_1 = \hat{T}\hat{b}_2' = \hat{T}\hat{S}_{22}'\hat{a}_2' = \hat{T}\hat{S}_{22}'\hat{T}\hat{b}_1. \tag{2-6}$$

The operator $\hat{T}\hat{S}_{22}'\hat{T}$ appearing here is the description of the receiving system, as a passive scattering object, referred to the reference plane S_1 of the transmitting antenna. Since this operator recurs frequently, we denote it briefly by $\hat{R}'$. Substituting (6) in (1), we obtain

$$\hat{b}_1 = \hat{S}_{10} a_0 + \hat{S}_{11}\hat{R}'\hat{b}_1, \tag{2-7}$$

which (at least when written out more fully) is seen to be an integral equation determining $\hat{b}_1$. (It may be identified as an inhomogeneous, linear integral equation of the second kind.) The solution may be indicated formally by writing

$$\hat{b}_1 = (\hat{1} - \hat{S}_{11}\hat{R}')^{-1}\hat{S}_{10} a_0. \tag{2-8}$$

This gives us the spectrum of outgoing waves in the transmission path; it includes both the simple plane waves and the evanescent waves. ($\hat{a}_1$ is now determined by (6); $\mathbf{E}_{1t}(\mathbf{r})$ and $\mathbf{H}_{1t}(\mathbf{r})$ are determined by (1.2-8).) We may obtain a more explicit but still formal solution to the basic integral

equation by the Liouville-Neumann method of successive substitutions. This leads to a representation of the inverse operator in (8) in a series of iterated operators,[9] so that

$$\hat{b}_1 = [\hat{1} + \hat{S}_{11}\hat{R} + (\hat{S}_{11}\hat{R})^2 + \ldots]\hat{S}_{10}a_0. \tag{2-9}$$

The special virtue of this form is that the successive terms in the series correspond to successive round-trip multiple reflections between the transmitting and the receiving antennas. Of course, (9) is meaningful as an infinite series only if it converges in some useful sense. The domain of convergence will depend upon the "smallness" of the product $\hat{S}_{11}\hat{R}'$, and it is worth noting that this product depends upon both $\hat{S}_{11}$ and $\hat{S}'_{22}$ (as well as upon the distance between the transducers).

We complete this analysis by calculating the scattering matrix of the "system 2-port," which has its terminals at S_0 and S'_0 and is defined by the equations

$$\begin{aligned} b_0 &= M_{11}a_0 + M_{12}a'_0, \\ b'_0 &= M_{21}a_0 + M_{22}a'_0. \end{aligned} \tag{2-10}$$

(The properties of a transmission system are often conveniently embodied in this form.) Inasmuch as we have made $a'_0 = 0$, solving for b_0/a_0 and for b'_0/a_0 yields directly

$$M_{11} = S_{00} + \hat{S}_{01}\hat{R}'(1 - \hat{S}_{11}\hat{R}')^{-1}\hat{S}_{10}, \tag{2-11}$$

$$M_{21} = \hat{S}'_{02}\hat{T}(1 - \hat{S}_{11}\hat{R}')^{-1}\hat{S}_{10}. \tag{2-12}$$

A similar alternative solution with $a_0 = 0$ and $a'_0 \neq 0$ yields

$$M_{22} = S'_{00} + \hat{S}'_{02}\hat{R}(1 - \hat{S}'_{22}\hat{R})^{-1}\hat{S}'_{20} \tag{2-13}$$

$$M_{12} = \hat{S}_{01}\hat{T}'(1 - \hat{S}'_{22}\hat{R})^{-1}\hat{S}'_{20}, \tag{2-14}$$

where $\hat{R} \equiv \hat{T}\hat{S}_{11}\hat{T}$. These formulas were first given in [41]; formally identical expressions are obtained in the electroacoustics case [3]. Their general significance was mentioned in the Introduction. Complete analytical solutions can be found in a highly specialized and idealized (but nevertheless interesting) class of problems; in one such solution the associated Liouville-Neumann series is incidentally found to converge or diverge according as kd is greater or less than 0.87993310 . . .

In the measurement technique to be described in subsection 3.1 (but not that in 3.2) we assume that the effects of reflections between antennas have been minimized and may be neglected. When such reflections are omitted, (12) and (13) become $M_{21} = \hat{S}'_{02}\hat{T}\hat{S}_{10}$ and $M_{22} = S'_{00}$, respectively. If the (passive) termination on the receiving antenna has reflection coefficient Γ_L, we obtain from (10)

$$b'_0 = F'\hat{S}'_{02}\hat{T}\hat{S}_{10}a_0,$$

where $F' = (1 - \Gamma_L S'_{00})^{-1}$. More explicitly, we have

$$b'_0 = F'a_0 \int \sum S'_{02}(m, \mathbf{K})S_{10}(m, \mathbf{K})e^{i\gamma d}d\mathbf{K}. \tag{2-15}$$

(Equation (15) is essentially a basic and simple case of eq (43) or (46) in [1].) The integral appearing in this equation is called the *transmission integral*, and the scalar product in the integrand,

[9] The operator expansion in (9) is analogous to the finite-dimensional matrix expansion $(1 - A)^{-1} = 1 + A + A^2 + A^3 + \ldots$, which is valid if all the eigenvalues of A are less than unity in magnitude.

$$\sum_m S'_{02}(m,\mathbf{K})S_{10}(m,\mathbf{K}) \equiv \mathbf{S}'_{02}(\mathbf{K})\cdot\mathbf{S}_{10}(\mathbf{K}), \tag{2-16}$$

is called the *coupling product*. This marks the emergence of the central quantities involved in the antenna measurement techniques described here.

In order to establish some of the content of (15) (but not for present applications) we note that the well-known Friis transmission formula can be derived from the asymptotic form of the equation. The asymptotic form in question is

$$b'_0 \sim -2\pi ikF'\mathbf{S}'_{02}(0)\cdot\mathbf{S}_{10}(0)a_0e^{ikd}/d. \tag{2-17}$$

(This is analytically a version of (1.2−16a) or (1.2−16b) evaluated on-axis.) If we now calculate the ratio of the available power at the receiving antenna terminals to the net power input at the transmitting antenna terminals and use the Schwarz inequality, we obtain

$$\frac{P'_A}{P_0} \leqslant \frac{G_1(0)\sigma'_2(0)}{4\pi d^2}, \tag{2-18}$$

where $G_1(0)$ pertains to the transmitting antenna, $\sigma'_2(0)$ pertains to the receiving antenna, and we have used (1.6−6, −19). Equality in (18) holds for polarization match.

We mention one more important result, contained in (11). The first iterated integral in the Liouville-Neumann series for M_{11} is the *reflection integral*,

$$\Phi(d) = \int d\mathbf{K}e^{i\gamma(K)d}\mathbf{S}_{01}(\mathbf{K})\cdot\int \mathbf{S}'_{22}(\mathbf{K},\mathbf{L})\cdot\mathbf{S}_{10}(\mathbf{L})e^{i\gamma(L)d}d\mathbf{L}. \tag{2-19}$$

This is the simplest form of integral involving a scattering process. It can be interpreted as a monostatic radar equation, which, apart from multiple reflections between target and transceiver, is valid at arbitrary distances. Examples of the use of this equation may be found in [1]. In particular, upon setting $\mathbf{S}'_{22}(\mathbf{K},\mathbf{L}) = -\mathbf{1}\delta(\mathbf{K}-\mathbf{L})$, where $\mathbf{1}$ denotes the transverse unit dyadic, one obtains the interferometer reflection-integral of reference [1].

3. Determination of Coupling-Product Values

3.1. Deconvolution of Transverse Scanning Data; Application of Sampling Theorem

We can now quite easily give the analytical basis for determining coupling-product values from transmission data taken in a transverse plane. Let the required relative transverse displacement between the transmitting and the receiving antennas be denoted by a transverse displacement $\mathbf{P} = x\mathbf{e}_x + y\mathbf{e}_y$ of the receiving antenna (fig. 8). By considering the phase $\mathbf{k}\cdot\mathbf{r}$ $(\mathbf{r} = \mathbf{P} + d\mathbf{e}_z)$ of the waves in the spectrum incident on the reference plane S'_2 of the receiving antenna, we see that the process of displacement simply introduces the factor $\exp(i\mathbf{K}\cdot\mathbf{P})$ in the integrand of (2−15). The expression for the received signal becomes

$$b'_0(\mathbf{P}) = a_0F'\int e^{i\mathbf{K}\cdot\mathbf{P}}\mathbf{S}'_{02}(\mathbf{K})\cdot\mathbf{S}_{10}(\mathbf{K})e^{i\gamma d}d\mathbf{K}. \tag{3.1-1}$$

Note that the quantity $b'_0(\mathbf{P})$ is what is observed in the measurement process; it may or may not be simply related to $\mathbf{E}$ at the point $(\mathbf{P}, d)$. Inasmuch as (1) represents a Fourier integral transformation, its inversion is immediate: We write

$$\mathbf{S}'_{02}(\mathbf{K})\cdot\mathbf{S}_{10}(\mathbf{K}) = D(\mathbf{K}), \tag{3.1-2}$$

where $D(\mathbf{K})$ is here an abbreviation for the determinate function of $\mathbf{K}$ given by

$$D(\mathbf{K}) \equiv \frac{e^{-i\gamma d}}{4\pi^2 F' a_0} \int b_0'(\mathbf{P}) e^{-i\mathbf{K}\cdot\mathbf{P}} d\mathbf{P}. \tag{3.1-3}$$

The inversion of (1) is appropriately termed deconvolution (each of the factors $\mathbf{S}_{02}'$ and $\mathbf{S}_{10}$ can be interpreted as Fourier transforms of certain physical fields). The term distinguishes the inversion of (1) from the inversion of the simpler equation (1.3–6) representing the Fourier-transform definition of $\mathbf{S}_{q0}$.

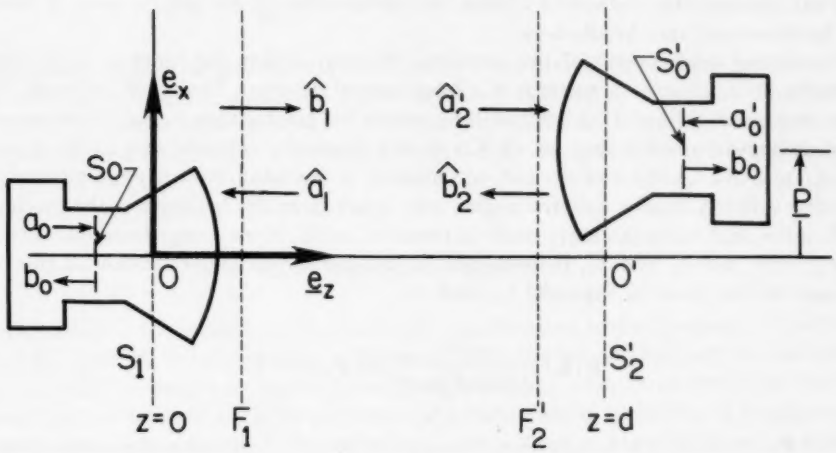

FIGURE 8. *Arrangement for transverse scanning: d is fixed,* **P** *is variable.*

The right-hand side of (3) is in fact determined (up to a phase factor) by the following measurable quantities: the distance d between the reference planes S_1 and S_2'; the reflection coefficients Γ_L and S_{00}'; and the magnitude and relative phase of $b_0'(\mathbf{P})/a_0$ as a function of **P**. (For convenience in measurement $b_0'(\mathbf{P})$ may be normalized in two stages, indicated by the expression

$$\frac{b_0'(\mathbf{P})}{a_0} = \frac{b_0'(\mathbf{P}_0)}{a_0} \cdot \frac{b_0'(\mathbf{P})}{b_0'(\mathbf{P}_0)},$$

where $\mathbf{P}_0$ is a selected fixed point.)

An instructive hypothetical case in which our analysis correctly shows that no probe correction, other than a known constant multiplier, would be needed occurs when the receiving antenna is considered to be an ideal electric-field probe. Such a probe can be described and treated analytically as an elementary electric dipole antenna, assumed lossless and reciprocal. For such a probe it can be shown that the receiving response is $b_0'(\mathbf{P}) = C\mathbf{E}(\mathbf{P}) \cdot \mathbf{e}_p$, where $\mathbf{e}_p$ is a unit vector giving the orientation of the dipole and the absolute value of the constant C can be calculated. Evidently the assumed use of an ideal probe oriented in the x-direction, say, in (3), would effectively reduce (3) to the $\mathbf{e}_x$-component of (1.3–6).

As (1) shows, planar scanning (in the absence of multiple reflections) can be interpreted rigorously as a spatially invariant, linear, filtering process in the two-dimensional wavenumber domain (for filtering concepts see e.g., [53]). The concept of the action of a receiving antenna as a filter has been noted, e.g., by Brown [19], apparently with non-planar scanning in mind. The processes are substantially different in the planar and the non-planar cases, as becomes clearly evident upon consideration of the action of a highly directive receiving antenna. Nevertheless, the ability to account fully for the probe characteristics in the planar case suggests the use of a well-charac-

terized "large" probe to reduce data taking and processing effort. Measurements at NBS using such a probe have produced the expected beneficial results. The filtering process has also been studied by Joy and Paris [10] using digital filtering (after data taking) to simulate probe characteristics.

Clearly the utility of the result (3) depends upon one's ability to evaluate the transform of the empirically observed $b_0'(\mathbf{P})$. Both least-square fitting and a two-dimensional form of the sampling theorem have been successfully used to evaluate the required transform of $b_0'(\mathbf{P})$ from data taken at the points of a rectangular lattice in the measurement plane [2, 8]. The application of the sampling theorem has become the method of choice, mainly because of the greater ease of computation, and will be described very briefly here.

The essential requirement of the sampling theorem is that the function to be sampled be representable as the Fourier transform of a band-limited function. That $b_0'(\mathbf{P})$ virtually fulfills this condition may be seen from (1): a band-limit K_B somewhat greater than k, and a distance d, may be chosen so that evanescent waves for all $K > K_B$ are assuredly virtually zero in the measurement plane (e.g., with $K_B = 1.05k$ and $d = 15\lambda$, attenuation at the band limit is approximately 260 dB). Bandlimiting within a smaller spectral region may result from the behavior of the product $\mathbf{S}_{02}'(\mathbf{K}) \cdot \mathbf{S}_{10}(\mathbf{K})$ in individual cases (an early study is reported in [8]). If we assign band limits $k_1 \equiv \pm 2\pi/\lambda_1$ and $k_2 \equiv \pm 2\pi/\lambda_2$ for k_x and k_y, respectively, a straightforward generalization of the usual one-dimensional theory, given in appendix C, leads to

$$D(\mathbf{K}) = \frac{e^{-i\gamma d}}{4k_1 k_2 F' a_0} \sum_{r,s} b_0'(\mathbf{P}_{rs}) e^{-i\mathbf{K}\cdot\mathbf{P}_{rs}} \tag{3.1-4}$$

The vectors $\mathbf{P}_{rs} = \frac{1}{2}r\lambda_1 \mathbf{e}_x + \frac{1}{2}s\lambda_2 \mathbf{e}_y$ (with $r, s = \ldots -1, 0, 1, 2, \ldots$) define the measurement lattice, the quantities $b_0'(\mathbf{P}_{rs})$ are the (complex) values of probe output directly observed at the points of the lattice, and the summation goes over the points of the lattice. According to the sampling theorem, (4) is mathematically exact; that is, if the data [the $b_0'(\mathbf{P}_{rs})$] were complete and exact, the result would be exact. (Although (4) is exact, it is not the "best possible" result; more advanced theory [42] shows that a rhomboidal lattice would be somewhat more efficient than the rectangular lattice.) The theorem requires an infinite sum, but in the applications thus far we have found that not even all values measurable above noise are needed.

An important feature of (4) is that the highly efficient algorithm known as the "fast Fourier transform" is rigorously applicable to evaluate the sum.

3.2. Received Signal as a Function of Distance and the Extrapolation Technique

The extrapolation technique was introduced by Wacker and Bowman, has been described very briefly by Newell and Kerns [9], and more fully, in an experimentally oriented paper, by Newell et al. [11]. The theory and numerical techniques were developed by Wacker [6]. The papers mentioned should be consulted for details; in the following paragraphs we give only a brief account of the main ideas and equations involved.

The extrapolation technique requires that one observe b_0'/a_0 as a function of antenna separation distance d, which is precisely defined by the choice of reference surfaces S_1 and S_2' associated with the respective antennas (fig. 7).

From (2–10) for the system 2-port, we obtain the expression

$$b_0' = a_0 M_{21}/(1 - M_{22}\Gamma_L), \tag{3.2-1}$$

which is a precise and complete version of (2–15). Expressions for the elements M_{21} and M_{22} are given in (2–12, –13), from which, by a rather lengthy process, one finds for (1) as a function of d a series representation of the form

$$b_0' = a_0 F' \sum_{p=0}^{\infty} \frac{\exp\left[i(2p+1)kd\right]}{d^{2p+1}} \sum_{q=0}^{\infty} A_{pq} d^{-q}. \tag{3.2-2}$$

We observe that the subseries of terms with a given p can be interpreted as the contribution of energy which has experienced $2p$ reflections or made $2p+1$ transits between antennas. In particular, the subseries with $p=0$ involves no reflections and is the expansion of the transmission integral (2–15):

$$\Psi(d) = \frac{e^{ikd}}{d}\left(A_{00} + A_{01} d^{-1} + A_{02} d^{-2} + \ldots\right). \tag{3.2-3}$$

It is of considerable analytical interest that this series is not merely asymptotic but actually convergent for sufficiently large d, under the main hypothesis that the two antennas involved be of finite size. If d is measured between centers of spheres, of radii r and r', each circumscribing one of the antennas, then $d > d_0 = r + r'$ is sufficient. A generally sharper, but more complicated, prescription for d_0 can be given [6, 7].

By comparison of (3) and (2–17) we see that

$$A_{00} = -2\pi i k \mathbf{S}_{02}'(0) \cdot \mathbf{S}_{10}(0). \tag{3.2-4}$$

Hence determination of the leading coefficient in (2) is tantamount to the determination of the (on-axis) value of the spectral coupling product. The basic idea of what we may call the conventional measurement method is simply to have d large enough to make other terms negligible compared to the leading term of the series. The basic idea of the extrapolation technique is to observe b_0' as a function of d and to fit this function with as many terms of (2) as may be significant, and so to determine a good value for A_{00} in particular. This enables one to cope with proximity effects and with multiple reflections between antennas.

4. Utilization of Coupling-Product Data

4.1. One Unknown Antenna (Transmitting or Receiving)

In this subsection we provide basic equations for the use of coupling-product data in a basic antenna measurement situation: the measurement of an unknown antenna (transmitting or receiving) with the requisite known antennas or antenna. Some of the concepts and notation established will be used in the next two subsections (which provide a partial answer to the question, "How does one obtain the first known antenna?").

In this subsection "nothing more" than the algebra of two (complex) equations in two (complex) unknowns is involved. The only likely case involving non-uniqueness and compatibility conditions is discussed. (In the following subsections the linear equations have to be reinterpreted as quadratics.)

It will suffice to consider only the case in which the unknown antenna is transmitting; the discussion of the other case would, of course, be analytically very similar.

We do not make simplifying a priori assumptions concerning symmetry or polarization (or other) characteristics of the antenna to be measured. Quite generally, then, we require measurements to be made with (at least in effect) *two* receiving antennas, A and B, having suitable known receiving characteristics, $\mathbf{A}_{02}(\mathbf{K})$ and $\mathbf{B}_{02}(\mathbf{K})$. From such measurements, the values of the coupling products

$$\begin{aligned} \mathbf{A}_{02}(\mathbf{K}) \cdot \mathbf{S}_{10}(\mathbf{K}) &= D_A(\mathbf{K}), \\ \mathbf{B}_{02}(\mathbf{K}) \cdot \mathbf{S}_{10}(\mathbf{K}) &= D_B(\mathbf{K}) \end{aligned} \tag{4.1-1}$$

are to be determined for the desired values of **K**. For each chosen, fixed value of the parameter **K**, we have two (complex) equations for the two (complex) components of $\mathbf{S}_{10}(\mathbf{K})$.

Complete solvability of (1) requires that the vectors $\mathbf{A}_{02}$ and $\mathbf{B}_{02}$ be linearly independent at the value of **K** considered. A measure of the linear independence of the two vectors is given by the following expression for the normalized squared-magnitude of the determinant of (1):

$$\frac{|\Delta|^2}{|\mathbf{A}|^2|\mathbf{B}|^2}=1-\frac{|\mathbf{A}\cdot\overline{\mathbf{B}}|^2}{|\mathbf{A}|^2|\mathbf{B}|^2}. \tag{4.1-2}$$

As the Schwarz inequality shows, this quantity ranges from the value zero when $\mathbf{B}_{02}$ is proportional to $\mathbf{A}_{02}$ to a maximum of unity when $\overline{\mathbf{B}}_{02}\cdot\mathbf{A}_{02}$ vanishes. In other words, power orthogonality represents the extreme case of linear independence. Examples are linear polarizations at right angles and left- and right-circular polarizations.

An explicit, coordinate-free solution of (1) may be obtained with the use of the set of vectors reciprocal to $\mathbf{A}_{02}$ and $\mathbf{B}_{02}$. The reciprocal set $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\alpha}\cdot\mathbf{A}_{02}=1,\qquad \boldsymbol{\alpha}\cdot\mathbf{B}_{02}=\boldsymbol{\beta}\cdot\mathbf{A}_{02}=0,\qquad \boldsymbol{\beta}\cdot\mathbf{B}_{02}=1$$

(assuming the required linear independence), and

$$\mathbf{S}_{10}=D_A\boldsymbol{\alpha}+D_B\boldsymbol{\beta} \tag{4.1-3}$$

as is easily verified. The algebra is summed up in the statement that D_A and D_B are the covariant components of $\mathbf{S}_{10}$ with respect to $\mathbf{A}_{02}$ and $\mathbf{B}_{02}$ as base vectors.

In much of what follows the use of x, y components relative to the fixed basis $\mathbf{e}_x$, $\mathbf{e}_y$—rather than the 1, 2 components relative to the variable unit vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$—is indicated. The consideration of geometric rotations and symmetries is appreciably complicated by the dependence of the $\boldsymbol{\kappa}$'s upon **K**. Also, the use of $\mathbf{e}_x$, $\mathbf{e}_y$ automatically takes care of the matter of defining the $\boldsymbol{\kappa}$'s on-axis. The required coordinate transformations will be governed by the relationship of the unit vectors

$$\boldsymbol{\kappa}_1=c\mathbf{e}_x+s\mathbf{e}_y,\qquad \boldsymbol{\kappa}_2=-s\mathbf{e}_x+c\mathbf{e}_y, \tag{4.1-4}$$

where $c\equiv\cos\phi=k_x/K$ and $s\equiv\sin\phi=k_y/K$.

We introduce the abbreviations

$$A_x=A_{02x}(\mathbf{K}),\qquad B_x=B_{02x}(\mathbf{K}),\qquad S_x=S_{10x}(\mathbf{K}),\qquad (x=x,y) \tag{4.1-5}$$

and write (1) in component form

$$\begin{aligned} A_xS_x+A_yS_y&=D_A,\\ B_xS_x+B_yS_y&=D_B. \end{aligned} \tag{4.1-6}$$

Instead of using two intrinsically different antennas A and B, in many cases it may be possible and convenient to use one antenna in two orientations, differing by rotation around the z axis by 90 degrees, say in the direction of x to y. If antenna A is so used and so rotated, we obtain for the equivalent of antenna B

$$\begin{aligned} B_{02x}(k_x,k_y)&=-A_{02y}(k_y,-k_x)\\ B_{02y}(k_x,k_y)&=A_{02x}(k_y,-k_x). \end{aligned} \tag{4.1-7}$$

These equations express the rotation of the vector field $\mathbf{A}_{02}$ corresponding to the rotation of the antenna that it describes.[10] They lead to a modified version of (6), which we shall discuss in the particularly interesting case of evaluation on-axis ($\mathbf{K}=0$). If we let $A_x^0=A_{02x}(0)$, $S_x^0=S_{10x}(0)$, etc., the modified form of (6) is

$$\begin{aligned} A_x^0 S_x^0 + A_y^0 S_y^0 &= D_A(0), \\ -A_y^0 S_x^0 + A_x^0 S_y^0 &= D_B(0). \end{aligned} \tag{4.1-8}$$

The determinant of the system (8) vanishes if and only if $A_x^0=\pm iA_y^0$; that is, the determinant vanishes if and only if the response characteristic $\mathbf{A}_{02}$ is "circularly polarized" at the point $\mathbf{K}=0$. This gives us a hint as to the special advantages of the use of circular polarization components for equations of the above form. We introduce circular polarization components for on-axis quantities in the following manner: For the transmitting characteristic

$$\mathbf{S}_{10}(0)=S_+^0\,\mathbf{e}_+ + S_-^0\mathbf{e}_- \tag{4.1-9a}$$

and for the receiving characteristic

$$\mathbf{A}_{02}(0)=A_+^0\,\bar{\mathbf{e}}_+ + A_-^0\,\bar{\mathbf{e}}_-, \tag{4.1-9b}$$

where the superposed bar denotes the complex conjugate (as usual) and

$$\mathbf{e}_+ \equiv (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}, \qquad \mathbf{e}_- \equiv (\mathbf{e}_x - i\mathbf{e}_y)/\sqrt{2}.$$

(The ordered pair of vectors $\bar{\mathbf{e}}_+$, $\bar{\mathbf{e}}_-$ in (9b) is reciprocal to the ordered pair $\mathbf{e}_+$, $\mathbf{e}_-$ in (9a).) The formulas for transformation to circular components are thus determined as

$$S_x^0=(S_+^0+S_-^0)/\sqrt{2}, \qquad S_y^0=i(S_+^0-S_-^0)/\sqrt{2}, \tag{4.1-10a}$$

$$A_x^0=(A_+^0+A_-^0)/\sqrt{2}, \qquad A_y^0=-i(A_+^0-A_-^0)/\sqrt{2}. \tag{4.1-10b}$$

The scalar products in (8) become

$$A_+^0 S_+^0 + A_-^0 S_-^0 = D_A(0), \tag{4.1-11a}$$

$$iA_+^0 S_+^0 - iA_-^0 S_-^0 = D_B(0). \tag{4.1-11b}$$

As is apparent, we have chosen the notation so that A_+ and A_- represent receptivities to the correspondingly labeled circular components of $\mathbf{S}_{10}$. The coupling is of course consistent with the polarization matching theorem (1.6-18). Equations (11) show clearly that if the receiving antenna were to respond to only one circular component of polarization, then that component, but not the other, could still be measured using one or the other of the two equations. If both equations were to be used, the compatibility condition (required by the vanishing of the determinant) should be satisfied within experimental error.

4.2. Generalized Two-Identical-Antenna Techniques

The technique to be described is formulated for nonreciprocal antennas, assuming that one possesses the adjoint of the antenna to be measured. If the antenna to be measured is reciprocal

[10] Concepts, analytical tools, and notation for the rotation of vector and tensor fields are discussed on p. 272 of [43].

(≡ self-adjoint), then the assumption is that one possesses duplicate antennas.[11] This is no doubt the more likely case but the more general formulation can be given with essentially no extra algebraic complication. Under the main assumption we are assured that the receiving characteristic of one antenna will be related to the transmitting characteristic of the other by reciprocity relations whether or not the antennas are reciprocal.

Granted the assumption of duplicate or mutually adjoint antennas, no additional assumptions are required to permit formulation and solution of equations for on-axis gain and polarization characteristics of both antennas. Additional a priori information, ordinarily qualitative, is required only for resolution of square-root sign ambiguities.

Certain commonly occurring types of symmetry permit one additionally to obtain solutions for off-axis values of gain and polarization.

For definiteness we assume transmission from left to right (as usual); limit the discussion to the determination of the right-side characteristics of the two (in general distinct) antennas labeled S and S^a, say; and choose to formulate equations for the direct determination of transmitting characteristics, leaving receiving characteristics to be determined by reciprocity. Under these ground rules, the remaining problem consists of at most two parts:

(a) Transmit from S to S^a; formulate equations for $\mathbf{S}_{1o}$, find $\mathbf{S}^a_{o1}$ by reciprocity;
(b) Transmit from S^a to S; formulate equations for $\mathbf{S}^a_{1o}$, find $\mathbf{S}_{o1}$ by reciprocity.

If reciprocity (in the ordinary sense) applies, the superscript "a" is without effect and may be omitted; the two cases reduce to one.

Inasmuch as the algebraic problem is in all cases substantially identical, it will be sufficient to consider only case (a) explicitly.

Both antennas are initially to be described in the same orientation and position relative to the fixed coordinate system $0xyz$. When one of the antennas is placed and oriented to serve in reception, its description relative to fixed coordinates will be changed accordingly. Indeed, the phase factor $\exp(i\gamma d)$ introduced by the axial translation can be regarded either as a modification of the receiving characteristic of the receiving antenna or as a property of the transmission path. We make the latter point of view explicit by referring the description of the receiving antenna to the shifted coordinate system $0'xyz'$, where $0'$ is at point $(0,0,d)$ in the original system $0xyz$ (fig. 7).

We shall need to consider the receiving antenna in two receiving orientations, differing by a 90 degree rotation around the z-axis. The operative characteristics of the receiving antenna in these two orientations will be distinguished by single and double primes.

Let the adjoint antenna be rotated into the first receiving orientation. This requires 180 degrees rotation around a transverse axis, say the y-axis. The operative receiving characteristic of the rotated antenna is then

$$\mathbf{S}^{a'}_{02}(\mathbf{K}) = P_{y2}\mathbf{S}^a_{01}(\mathbf{K}), \tag{4.2-1}$$

where P_{y2} is a notation for the transformation (of $\mathbf{S}^a_{01}$) produced by the prescribed rotation. Now the transformed function $P_{y2}\mathbf{S}^a_{01}$ is related by reciprocity, (1.5-4), to the similarly transformed function $P_{y2}\mathbf{S}_{10}$. Thus, if the reciprocity relation is written in vector form, we have

$$\eta_0\mathbf{S}^{a'}_{02}(\mathbf{K}) = \boldsymbol{\eta}\cdot P_{y2}\mathbf{S}_{10}(-\mathbf{K}); \tag{4.2-2}$$

and if it is written in x, y component form we have

[11] The idea of using a reflecting surface or mirror to produce an image antenna is not fully applicable, even if the antenna is reciprocal: Coupling-product data can be obtained by extrapolation or by conventional techniques, but not by transverse scanning. The coupling-product equations can be formulated and solved for on-axis values provided on-axis polarization is known and not circular. It should be noted that the image antenna, being a mirror image, cannot in general be considered an identical antenna even with a perfectly reflecting surface of infinite area.

$$\eta_0\begin{pmatrix} S'_{02x}(\mathbf{K}) \\ S'_{02y}(\mathbf{K}) \end{pmatrix}=\begin{pmatrix} \eta_1c^2+\eta_2s^2 & (\eta_1-\eta_2)cs \\ (\eta_1-\eta_2)cs & \eta_1s^2+\eta_2c^2 \end{pmatrix}\begin{pmatrix} -S_{10x}(k_x,-k_y) \\ S_{10y}(k_x,-k_y) \end{pmatrix} \tag{4.2-3}$$

Here the square matrix is determined by (4.1-4) and the column matrix on the right contains the x, y components of $P_{y2}\mathbf{S}_{10}(-\mathbf{K})$. We may now evaluate the coupling product $\mathbf{S}_{10}(\mathbf{K})\cdot\mathbf{S}'_{02}(\mathbf{K})\equiv D'(\mathbf{K})$. Some degree of abbreviation is indispensable; we use

$$S_x=S_{10x}(\mathbf{K}),\quad S_y=S_{10y}(\mathbf{K}),\quad S^v_x=S_{10x}(k_x,-k_y),\quad S^v_y=S_{10y}(k_x,-k_y), \tag{4.2-4}$$

and obtain

$$-(\eta_1c^2+\eta_2s^2)S_xS^v_x+(\eta_1-\eta_2)scS_xS^v_y+$$

$$-(\eta_1-\eta_2)scS_yS^v_x+(\eta_1s^2+\eta_2c^2)S_yS^v_y=\eta_0D'(\mathbf{K}). \tag{4.2-5}$$

This is the first of the desired "measurement equations" relating the mathematical expression of the coupling product to its empirically determined values. It is interesting that the expression is invariant with respect to the interchange of k_y and $-k_y$; the empirical $D'(\mathbf{K})$ should also have this symmetry. Further, since $D'(\mathbf{K})$ is related to $b'_0(\mathbf{P})$ by (3.1-1), the $b'_0(\mathbf{P})$ data should have the corresponding property of invariance with respect to the interchange of y and $-y$. These general constraints should be experimentally useful.

To obtain the second measurement equation, we rotate the receiving antenna, as described by (1), 90 degrees around the z-axis in the direction x to y (cf. (4.1-7)). Using the notation P_{z4} for this rotation and applying the reciprocity relation, as in (2), we obtain

$$\eta_0\mathbf{S}''_{02}(\mathbf{K})=\eta\cdot P_{z4}P_{y2}\mathbf{S}_{10}(-\mathbf{K}). \tag{4.2-6}$$

In x, y component form this is

$$\eta_0\begin{pmatrix} S''_{02x}(\mathbf{K}) \\ S''_{02y}(\mathbf{K}) \end{pmatrix}=\begin{pmatrix} \eta_1c^2+\eta_2s^2 & (\eta_1-\eta_2)sc \\ (\eta_1-\eta_2)sc & \eta_1s^2+\eta_2c^2 \end{pmatrix}\begin{pmatrix} -S_{10y}(k_y,k_x) \\ -S_{10x}(k_y,k_x) \end{pmatrix} \tag{4.2-7}$$

Using the abbreviations

$$S^d_x=S_{10x}(k_y,k_x),\qquad S^d_y=S_{10y}(k_y,k_x), \tag{4.2-8}$$

as well as S_x and S_y in (4), we find for the coupling product $S_{10}(\mathbf{K})\cdot\mathbf{S}''_{02}(\mathbf{K})\equiv D''(\mathbf{K})$ the expression

$$-(\eta_1c^2+\eta_2s^2)S_xS^d_y-(\eta_1-\eta_2)scS_xS^d_x+$$

$$-(\eta_1-\eta_2)scS_yS^d_y-(\eta_1s^2+\eta_2c^2)S_yS^d_x=\eta_0D''(\mathbf{K}). \tag{4.2-9}$$

In this case the coupling product is invariant with respect to the interchange of k_x and k_y, and again this constraint should be experimentally useful.

In the remainder of this subsection we discuss briefly some conditions and methods for determining components of $\mathbf{S}_{10}$ from (5) and (9), assuming that the D's are given for the values of $\mathbf{K}$ of interest. When evaluated for $\mathbf{K}\neq 0$, the six quantities S_x, S_y, S^v_x, S^v_y, S^d_x, and S^d_y appearing

in these two equations are in general distinct and unknown; obviously, some specialization or additional data are required. We consider two cases: evaluation on-axis, and a simple type of symmetry.

(i) *Evaluation on axis*—Evaluated on-axis, the six unknowns reduce to two

$$S_x^0 = S_{10x}(0), \qquad S_y^0 = S_{10y}(0), \tag{4.2-10}$$

and the coupling-product equations reduce to

$$-(S_x^0)^2 + (S_y^0)^2 = D'(0)\eta_0/Y_0, \tag{4.2-11a}$$

$$-2S_x^0 S_y^0 = D''(0)\eta_0/Y_0. \tag{4.2-11b}$$

(The key to this reduction is the observation that Y_0 is the common value of η_1 and η_2 on-axis.) Subcases under this case occur if the polarization on-axis is considered known. One or the other of (11) will suffice, no matter what that polarization may be.

Suppose, for example, that $\rho = S_y^0/S_x^0$ is considered known and not equal to ± 1; then from (11a) we may obtain

$$S_x^0 = \left(\frac{\eta_0 D'(0)}{Y_0(\rho^2 - 1)}\right)^{1/2}. \tag{4.2-12}$$

This, together with $S_y^0 = \rho S_x^0$, gives us the on-axis pattern vector in terms of $D'(0)$ (up to a sign). From the expression (1.6–6) for power gain we find

$$G_1(0) = \frac{4\pi k^2(|\rho|^2 + 1)|D'(0)|}{(1 - |S_{00}|^2)|\rho^2 - 1|}. \tag{4.2-13}$$

We can obtain an interesting form for this result by expressing $D'(0)$ in terms of the integral of $b_0'(\mathbf{P})$, as in the deconvolution relation, (3.1–3). Thus,

$$G_1(0) = \frac{k^2}{\pi}\frac{|1 - \Gamma_L' S_{00}'|}{1 - |S_{00}|^2}\frac{|\rho|^2 + 1}{|\rho^2 - 1|}\frac{1}{|a_0|}\left|\int b_0'(\mathbf{P})\, d\mathbf{P}\right|. \tag{4.2-14}$$

(This result, as well as the version of it with $\rho = 0$, has been presented previously [41, 44].) The essential simplicity of the result is somewhat obscured by the presence of the mismatch factors. If we assume a polarization match ($\rho = 0, \pm i$, or ∞), a conjugate impedance match ($\Gamma_L' = \bar{S}_{00}'$), and $S_{00} = S_{00}'$ (as appropriate for mutually adjoint or for identical antennas), we have

$$G_1(0) = \frac{k^2}{\pi}\cdot\frac{1}{|a_0|}\left|\int b_0'(\mathbf{P})\, dP\right|.$$

For the effective area, using (1.6 − 21b), we obtain the remarkably simple expression

$$\sigma_1^a(0) = \frac{1}{|a_0|}\left|\int b_0'(\mathbf{P})\, d\mathbf{P}\right|. \tag{4.2-15}$$

Equation (13) has been successfully applied experimentally [8], and (15) has been tested analytically in a special case. In the analytical test the two identical antennas were taken to be x-oriented elementary electric-dipole antennas, assumed lossless and reciprocal. In this case one does indeed obtain the expected result $\sigma_1(0) = 3\lambda^2/(8\pi)$.

More generally, the polarization is not known and it is necessary to solve (11) as simultaneous quadratics. A solution in terms of circular polarization components is convenient and useful.

Using the definitions in (4.1–9a), one obtains

$$(S_+^0)^2 = -\frac{\eta_0}{2Y_0}\,[D'(0) - iD''(0)],$$
$$(S_-^0)^2 = -\frac{\eta_0}{2Y_0}\,[D'(0) + iD''(0)]. \tag{4.2–16}$$

From these equations we obtain four pairs of values for the x, y components:

$$S_x^0 = \pm\frac{i}{2}\sqrt{\frac{\eta_0}{Y_0}}\,(\sqrt{D'(0) + iD''(0)} \pm \sqrt{D'(0) - iD''(0)}\,).$$
$$S_y^0 = \pm\frac{1}{2}\sqrt{\frac{\eta_0}{Y_0}}\,(\sqrt{D'(0) + iD''(0)} \mp \sqrt{D'(0) - iD''(0)}\,). \tag{4.2–17}$$

(The double signs are correlated vertically but not horizontally.) Which solution-pair pertains to a given measurement cannot be determined from the equations alone. It would seem that ordinarily the overall plus-minus signs should be of no significance. The remaining sign-choice does affect the determination of the polarization index $\rho = S_y^0/S_x^0$. If we write $\rho^{(u)}$ and $\rho^{(l)}$ for the values associated with the upper and the lower signs, respectively, we find

$$\rho^{(u)}\rho^{(l)} = -1. \tag{4.2–18}$$

With the aid of this equation a modicum of a priori information about the magnitude and/or the phase of ρ should ordinarily be sufficient to resolve the ambiguity. For example, if the antenna is known to be approximately linearly polarized (on-axis) in a certain direction (we may choose the y axis in that direction), then $|\rho|$ is distinctly greater than unity and one would choose $\rho^{(u)}$ or $\rho^{(l)}$ accordingly. However, if the polarization is nearly circular, the difference between the two indices becomes relatively small and the choice could be difficult.

The squared magnitude $|S_x^0|^2 + |S_y^0|^2$, which determines the on-axis power gain, is unambiguous. In fact, for this gain we find

$$G_1(0) = \frac{2\pi k^2\,[|D'(0) + iD''(0)| + |D'(0) - iD''(0)|]}{1 - |S_{00}|^2} \tag{4.2–19}$$

(ii) *Additional solutions permitted by symmetry.* Certain types of symmetry enable one to determine certain off-axis values of the unknown functions. We consider one type of symmetry of frequent occurrence, exemplified by the fields of (a) rectangular waveguide open-ended or with a pyramidal horn, fed by the TE_{10} mode in the waveguide; (b) circular waveguide open-ended or with a circularly symmetric horn, fed by the TE_{11} mode in the waveguide; and (c) a transverse electric dipole. In these examples the symmetry may be analytically specified in terms of spectral components by

$$S_{10x}(-k_x,k_y) = S_{10x}(k_x,-k_y) = -S_{10x}(k_x,k_y),$$
$$S_{10y}(-k_x,k_y) = S_{10y}(k_x,-k_y) = \quad S_{10y}(k_x,k_y). \tag{4.2–20}$$

(Here a specific orientation of structures and fields has of course been assumed.) We note in particular that S_{10x} must vanish on the coordinate axes. Hence, the four unknown functions involved in (5) are reduced to the single one, $S_{10y}(k_x,0)$ on the line $k_y = 0$, and to the single func-

tion $S_{10y}(0,k_y)$ on the line $k_x = 0$. From (5) we immediately obtain the separate equations determining these two functions,

$$\eta_2[S_{10y}(k_x,0)\,]^2 = \eta_0 D'(k_x,0)\,,$$
$$\eta_1[S_{10y}(0,k_y)\,]^2 = \eta_0 D'(0,k_y)\,. \tag{4.2-21}$$

Next we notice that on the diagonal line $k_y = k_x$, we have $S_x^d \equiv S_x$ and $S_y^d \equiv S_y$. Additionally, the symmetry furnishes $S_x = -S_x^v$ and $S_y = S_y^v$. Hence, of the six unknown functions involved in (5) and (9) essentially only two distinct ones survive,

$$u_x \equiv S_{10x}(k_x, k_x), \qquad u_y \equiv S_{10y}(k_x, k_x)\,,$$

say. Further, the two quadratics can easily be solved for these unknowns, as the following intermediate results show:

$$\eta_2(u_x - u_y)^2 = \eta_0[D'(k_x, k_y) + D''(k_x, k_x)]\,,$$
$$\eta_1(u_x + u_y)^2 = \eta_0[D'(k_x, k_x) - D''(k_x, k_x)\,]\,. \tag{4.2-22}$$

Hence in this case we can obtain solutions on the four lines, $k_x = 0, k_y = 0$, and $k_y = \pm k_x$.

4.3. Generalized Three-Antenna Techniques

In this subsection we discuss the analysis involved in techniques for determination of both power gain and polarization using three unknown (dissimilar) antennas. We require an antenna T, to be used only in transmitting; an antenna R, to be used only in reception; and an antenna S to be used in both receiving and transmitting modes. We do not need to inquire whether either of the antennas T and R is reciprocal or even capable of operating in a "reversed" mode. We do require either that antenna S be reciprocal or, if not reciprocal, capable of being "switched" to become its own adjoint S^a.

It is interesting and important that some kind of reciprocity[12] is indispensably required a priori information. This requirement cannot be avoided by increasing the number of antennas involved—even to the extent of using all possible combinations of n transmitting and m receiving antennas. But when the reciprocity requirement is met, as with a reciprocal or switchable antenna S, 3 antennas are sufficient. (Note that in the switchable case, S and its adjoint do not coexist. Actual simultaneous possession of both S and S^a would distinctly change the character of the measurement problem: In this case one could use the generalized 2-identical-antenna technique of the preceding subsection.)

Of course the use of 3 antennas in a measurement scheme is not in itself new. The use of 3 antennas in the roles of T, R, and S (S being reciprocal) is recognizable in a discussion in Vol. 12 of the MIT Radiation Laboratory Series—material originating in the early 1940's [45]. However, in that discussion simplifying assumptions regarding polarization were made and the analytical problem was reduced to the use of scalar equations involving gain only. Analogous schemes using three electroacoustic transducers coupled by a fluid medium are well known [46]. The application of the requisite electroacoustic reciprocity relations was introduced by Maclean [47] and by Cook [48] in 1940 and 1941, respectively.

In this discussion we consider only on-axis values of antenna characteristics and refer to these quantities as *the* antenna characteristics. Determination of off-axis characteristics by three-

[12] Conceivably some linear relation, other than that provided by ordinary or adjoint reciprocity, between receiving and transmitting characteristics might be known a priori. This would not in general lead to the relatively simple equations that are obtained under the assumption made.

antenna techniques has not yet been seriously considered, to our knowledge. (The corresponding electroacoustics problem, which is much simpler, has recently been completely solved under minimum assumptions in work as yet unpublished.)

When antenna polarization characteristics are fully taken into account, as in the present discussion, three-antenna techniques for the determination of on-axis characteristics require less a priori information than any other technique. Thus versions of the three-antenna technique have been the method of choice in several critical applications [e.g., 9, 11].

The problem now posed is taken to be the determination of the receiving characteristic $\mathbf{R}_{01}(0)$ of the antenna R, the transmitting characteristic $\mathbf{T}_{10}(0)$ of the antenna T, and the four "right-side" characteristics $\mathbf{S}_{10}(0)$, $\mathbf{S}_{01}(0)$, $\mathbf{S}^a_{10}(0)$, $\mathbf{S}^a_{01}(0)$ of the antennas S and S^a. We further choose to eliminate the receiving characteristics of S and S^a by means of the reciprocity relations, which for $\mathbf{K} = 0$ are simply

$$\eta_0 \mathbf{S}_{01}(0) = Y_0 \mathbf{S}^a_{10}(0), \qquad \eta_0 \mathbf{S}^a_{01}(0) = Y_0 \mathbf{S}_{10}(0). \tag{4.3-1}$$

In this discussion we use the notation

$$R_x = R_{02x}(0), \qquad S_x = S_{10x}(0), \qquad S^a_x = S^a_{10x}(0), \qquad T_x = T_{10x}(0) \qquad (x = x, y) \tag{4.3-2}$$

for the components of the four remaining vector unknowns. Certainly, to determine these 8 (complex) quantities we need a system of 8 (complex) equations; the "core" of the problem, however, turns out to be the solution of six simultaneous quadratic equations for six of the unknowns. (This is the complete solution if antenna S is reciprocal.) The remainder of the problem requires only the solution of two linear equations in two unknowns; this part will be called the "supplementary" problem.

The complete problem requires coupling-product data derived from transmission between antennas paired as shown in table 2. Transmission is from left to right (as already implied in the notation), and for each antenna-pair considered, the receiving antenna is used in two orientations differing by 90 degree rotation around the z axis. The first two columns in the table indicate two ways of utilizing the same data and are related by the interchange of the roles of S and S^a. If the antenna S is reciprocal ($S = S^a$), the first two columns coincide and reduce to the third; only the core problem remains. It will be sufficient to formulate and solve the algebraic problem represented in the first column of the table; the corresponding formulations and solutions may be obtained for the second column by interchanging S and S^a and for the case of reciprocal S by eliminating the superscript "a" (and with it the equations thus rendered superfluous).

TABLE 2. *Antenna pairings*

	$S \neq S^a$		$S = S^a$
For "core" problem:	$T \to R$ $S \to R$ $T \to S^a$	$T \to R$ $S^a \to R$ $T \to S$	$T \to R$ $S \to R$ $T \to S$
For "supplementary" problem:	$S^a \to R$	$S \to R$	–

For the transmission from T to R, we have

$$T_x R_x + T_y R_y = D'_{RT}, \qquad -T_x R_y + T_y R_x = D''_{RT}. \tag{4.3-3}$$

These equations are an instance of (4.1–8) (except that here characteristics of both antennas are unknown). For the transmission from S to R, we have

$$S_x R_x + S_y R_y = D'_{RS}, \qquad -S_x R_y + S_y R_x = D''_{RS}. \tag{4.3-4}$$

These equations are similarly an instance of (4.1–8). For the transmission from T to S^a, we have

$$-T_xS_x + T_yS_y = \eta_0 D'_{ST}/Y_0, \qquad -T_xS_y - T_yS_x = \eta_0 D''_{ST}/Y_0. \tag{4.3–5}$$

The receiving antenna, S^a, is here treated in the same way as was the receiving antenna in the two-antenna technique described in the preceding subsection (cf. (4.2–10)). This completes the formulation of the "core" problem. The solution is conveniently accomplished with the aid of circular polarization components, which are defined as in (4.1–9). Equations (3), (4), and (5) transform by pair to

$$\left.\begin{aligned} T_-R_- &= \frac{1}{2}(D'_{RT} + iD''_{RT}) \equiv \Sigma_{RT}, \\ T_+R_+ &= \frac{1}{2}(D'_{RT} - iD''_{RT}) \equiv \Delta_{RT}; \end{aligned}\right\} \tag{4.3–6}$$

$$\left.\begin{aligned} S_-R_- &= \frac{1}{2}(D'_{RS} + iD''_{RS}) \equiv \Sigma_{RS}, \\ S_+R_+ &= \frac{1}{2}(D'_{RS} - iD''_{RS}) \equiv \Delta_{RS}; \end{aligned}\right\} \tag{4.3–7}$$

$$\left.\begin{aligned} T_+S_+ &= -\frac{1}{2}\eta_0(D'_{ST} - iD''_{ST})/Y_0 \equiv -\eta_0\Delta_{ST}/Y_0, \\ T_-S_- &= -\frac{1}{2}\eta_0(D'_{ST} + iD''_{ST})/Y_0 \equiv -\eta_0\Sigma_{ST}/Y_0. \end{aligned}\right\} \tag{4.3–8}$$

For the squares of the individual circular polarization components one finds without difficulty (but note that neither this set of equations nor the preceding sets can be generated by straightforward advancement of subscripts)

$$\left.\begin{aligned} R_-^2 &= -\frac{Y_0}{\eta_0}\frac{\Sigma_{RT}\Sigma_{RS}}{\Sigma_{ST}}, \ (T_-S_- \neq 0) \\ R_+^2 &= -\frac{Y_0}{\eta_0}\frac{\Delta_{RT}\Delta_{RS}}{\Delta_{ST}}; \ (T_+S_+ \neq 0) \end{aligned}\right\} \tag{4.3–9}$$

$$\left.\begin{aligned} S_+^2 &= -\frac{\eta_0}{Y_0}\frac{\Delta_{RS}\Delta_{ST}}{\Delta_{RT}}, \ (T_+R_+ \neq 0) \\ S_-^2 &= -\frac{\eta_0}{Y_0}\frac{\Sigma_{RS}\Sigma_{ST}}{\Sigma_{RT}}; \ (T_-R_- \neq 0) \end{aligned}\right\} \tag{4.3–10}$$

$$\left.\begin{aligned} T_+^2 &= -\frac{\eta_0}{Y_0}\frac{\Delta_{RT}\Delta_{ST}}{\Delta_{RS}}, \ (S_+R_+ \neq 0) \\ T_-^2 &= -\frac{\eta_0}{Y_0}\frac{\Sigma_{RT}\Sigma_{ST}}{\Sigma_{RS}}. \ (S_-R_- \neq 0) \end{aligned}\right\} \tag{4.3–11}$$

The inequalities shown in parentheses (which in effect prohibit circular polarizations) must be in force if indeterminacy in the respective associated equations is to be avoided. If circular polarizations do occur, then obviously many special cases are possible. For example, if the transmitting characteristic of antenna S is (very nearly) right circularly polarized ($S_y = iS_x$), then (very nearly) $S_- = 0$, $\Sigma_{RS} = 0$, $\Sigma_{ST} = 0$, and R_- and T_- become experimentally indeterminate. However, the determination of the other circular components of R and T, and the determination of the transmitting characteristics of antenna S itself, would present no special difficulty. In practice, if a par-

tial solution such as this is not sufficient, one might include additional antennas in the scheme. Antennas capable of being switched between right and left circular polarizations have been used for similar reasons, usually in less complicated circumstances.

It remains to dispose of what we called the supplementary problem earlier in this discussion. If antenna S is reciprocal, there is no supplementary problem; if S is not reciprocal, it remains to determine $\mathbf{S}^a_{\dagger 0}(0)$. As suggested in the first column of table 2, this can be done by measurement with the receiving antenna R, for which calibration is presumptively available as a result of the core problem. Equations (4.1–11) are applicable; in notation adapted to the present context those equations become

$$R_+ S^a_+ + R_- S^a_- = D'_{RS}{}^a$$

$$iR_+ S^a_+ - iR_- S^a_- = D''_{RS}{}^a.$$

Here complete solvability requires $R_+ R_- \neq 0$.

Effective areas and power gains for antennas R, S, S^a, and T (as pertinent) are readily expressed in terms of circular polarization components, which by reference to (4.1–9), are seen to be normalized so that

$$|A_x|^2 + |A_y|^2 = |A_+|^2 + |A_-|^2.$$

Thus, for example, we obtain for antenna S from (10) and (1.6–6)

$$G_1(0) = \frac{4\pi k^2}{1 - |S_{00}|^2}\left(\left| \frac{\Delta_{RS}\Delta_{ST}}{\Delta_{RT}} \right| + \left| \frac{\Sigma_{RS}\Sigma_{ST}}{\Sigma_{RT}} \right| \right). \tag{4.3–12}$$

The corresponding effective area is given by (1.6–21a) or (1.6–21b), as appropriate.

Algebraic values for the individual linear polarization components may be obtained from (9), (10), or (11), as desired. For example, for antenna S

$$\begin{aligned} S_x &= \pm\, i \sqrt{\frac{\eta_0}{Y_0}}\left(\sqrt{\frac{\Sigma_{RS}\Sigma_{ST}}{\Sigma_{RT}}} \pm \sqrt{\frac{\Delta_{RS}\Delta_{ST}}{\Delta_{RT}}} \right), \\ S_y &= \pm \sqrt{\frac{\eta_0}{Y_0}}\left(\sqrt{\frac{\Sigma_{RS}\Sigma_{ST}}{\Sigma_{RT}}} \mp \sqrt{\frac{\Delta_{RS}\Delta_{ST}}{\Delta_{RT}}} \right), \end{aligned} \tag{4.3–13}$$

where the double signs are correlated vertically but not horizontally. As far as sign ambiguities are concerned, these equations have the same algebraic structure as (4.2–17) and the earlier discussion is again applicable.

5. Appendices

5.1. Appendix A: Reciprocity Theorems

The domain of the electromagnetic fields under consideration is the source-free region V bounded externally by the surfaces F_1 and F_2 and internally by the closed surface $S_0 + S_a$, which encloses the source or detector associated with the antenna considered. (See sec. 1.1 and fig. 1.) A passive antenna is a scatterer (sometimes called a "loaded scatterer"); if the structure is merely a scatterer, the surface $S_0 + S_a$ is irrelevant and may be disregarded.

We shall write Maxwell's equations in a form especially suited to the purposes of the present discussion. We associate with the given system, described by the constitutive equations (1.5–1), the "Maxwellian" operator

$$M=\begin{pmatrix} i\omega\boldsymbol{\epsilon}\cdot & i\omega\boldsymbol{\tau}\cdot+\nabla\times \\ -i\omega\boldsymbol{\nu}\cdot+\nabla\times & -i\omega\boldsymbol{\mu}\cdot \end{pmatrix}$$

and place the vectors **E** and **H** in the column matrix

$$\Phi=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix},$$

so that Maxwell's equations for a possible field in the system are expressed by

$$M\Phi=0. \tag{A-1}$$

Using the constitutive equations (1.5–2) we similarly obtain for the Maxwellian operator associated with the adjoint system

$$\tilde{M}=\begin{pmatrix} i\omega\tilde{\boldsymbol{\epsilon}}\cdot & -i\omega\tilde{\boldsymbol{\nu}}\cdot+\nabla\times \\ i\omega\tilde{\boldsymbol{\tau}}\cdot+\nabla\times & -i\omega\tilde{\boldsymbol{\mu}}\cdot \end{pmatrix},$$

where $\tilde{M}$ is in a natural sense the transpose of the operator M. Maxwell's equations for the fields in the adjoint system take the form

$$\tilde{M}\Phi=0. \tag{A-2}$$

The adjoint system represents a possible physical embodiment of the mathematical concept of adjoint differential expressions, which may be defined as follows. Let

$$\Phi'=\begin{pmatrix}\mathbf{F}'\\ \mathbf{G}'\end{pmatrix},\quad \Phi''=\begin{pmatrix}\mathbf{F}''\\ \mathbf{G}''\end{pmatrix}, \tag{A-3}$$

where $\mathbf{F}'$, $\mathbf{G}'$, $\mathbf{F}''$, and $\mathbf{G}''$ are arbitrary differentiable vector functions defined in V. Then we can associate a unique adjoint differential expression M^a with M by requiring that

$$\tilde{\Phi}''\cdot M\Phi'-\tilde{\Phi}'\cdot M^a\Phi''\equiv D, \tag{A-4}$$

where D is a divergence expression, be an identity in Φ' and Φ''. This leads to

$$M^a=\tilde{M} \tag{A-5a}$$

and

$$D=\nabla\cdot(\mathbf{F}'\times\mathbf{G}''-\mathbf{F}''\times\mathbf{G}'). \tag{A-5b}$$

Thus the adjoint operator turns out to be the transpose.[13]

We may now easily obtain the basic theorem for our purposes. If we replace $\mathbf{F}'$, $\mathbf{G}'$ and $\mathbf{F}''$, $\mathbf{G}''$ by $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{E}''$, $\mathbf{H}''$, which satisfy (1) and (2), respectively, we have

$$\nabla\cdot(\mathbf{E}'\times\mathbf{H}''-\mathbf{E}''\times\mathbf{H}')=0 \tag{A-6}$$

throughout the region V. This is a generalization of the well-known Lorentz relation. It may very easily be modified to include the effects of both electric and magnetic current sources embedded in V [37]. Here, however, we are interested in excitation of the systems by means of incident waves

[13] To avoid possible confusion, we emphasize that the adjoint arising here is definable essentially by transposition (without complex conjugation); it is not the Hermitian adjoint that is frequently useful in other physical problems. For discussion of the theory of adjoint differential expressions see especially Lanczos [49] or (for ordinary differential equations) Courant-Hilbert [50]. Equation (5a) incidentally contains the nontrivial result that the operator $\nabla\times$ is self-adjoint.

(in space or waveguide). With or without current-source terms, (6) characterizes what may properly be called *adjoint reciprocity*.

If the constitutive tensors obey the symmetry relations

$$\boldsymbol{\epsilon}=\tilde{\boldsymbol{\epsilon}}, \qquad \boldsymbol{\mu}=\tilde{\boldsymbol{\mu}}, \qquad \text{and } \boldsymbol{\tau}=-\tilde{\boldsymbol{\nu}} \tag{A-7}$$

then (as may be seen from (1.5-1, -2)) the adjoint system and the original system are identical (and $M^a=\tilde{M}=M$). In this case we say that the linear differential operators and the systems are self-adjoint—and ordinary reciprocity obtains. Equation 6 still holds; $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{E}''$, $\mathbf{H}''$ may be interpreted as distinct electromagnetic fields in one and the same system.

To apply the generalized Lorentz relation, we first take the volume integral of the expression over the region V and use the divergence theorem. This yields

$$\int_{S_0} \boldsymbol{L}\cdot\mathbf{n}_0\, dS=\sum_q \int_{F_q} \boldsymbol{L}\cdot\mathbf{n}_q\, d\mathbf{R}, \tag{A-8}$$

where $\boldsymbol{L}=\mathbf{E}'\times\mathbf{H}''-\mathbf{E}''\times\mathbf{H}'$ and the unit normals are inward on S_0 and outward on F_q (as prescribed in subsections 1.1 and 1.2). Next we substitute the modal representations of the fields $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{E}''$, $\mathbf{H}''$ on S_0, F_1 and F_2, using single and double primes to distinguish the spectral variables associated with the respective fields. One obtains after some analysis the generalized reciprocity lemma

$$\eta_0(a_0''b_0'-a_0'b_0'')$$
$$=\sum_p \int_{\mathbf{L}} \sum_n \left[a_p''(n,-\mathbf{L})\,b_p'(n,\mathbf{L})-a_p'(n,\mathbf{L})\,b_p''(n,-\mathbf{L})\right]\eta_n(L)\,d\mathbf{L}. \tag{A-9}$$

From this lemma, by making suitable special choices of excitation, we shall obtain the desired set of relations between the elements of the scattering matrix of the original system and the corresponding elements of the scattering matrix of the adjoint system.

For immediate reference we write down the scattering equations for the original system

$$\begin{aligned} b_0' &= S_{00}a_0' + \sum_p \int_{\mathbf{L}} \sum_n S_{0p}(n,\mathbf{L})\; a_p'(n,\mathbf{L})\; d\mathbf{L}, \\ b_q'(m,\mathbf{K}) &= S_{q0}(m,\mathbf{K})\; a_0' + \sum_p \int_{\mathbf{L}} \sum_n S_{qp}(m,\mathbf{K};\,n,\mathbf{L})\; a_p'(n,\mathbf{L})\; d\mathbf{L}. \end{aligned} \tag{A-10}$$

and those for the adjoint system

$$\begin{aligned} b_0'' &= S^a_{00}a_0'' + \sum_p \int_{\mathbf{L}} \sum_n S^a_{0p}(n,\mathbf{L})\; a_p''(n,\mathbf{L})\; d\mathbf{L}, \\ b_q''(m,\mathbf{K}) &= S^a_{q0}(m,\mathbf{K})\; a_0'' + \sum_p \int_{\mathbf{L}} \sum_n S^a_{qp}(m,\mathbf{K};\,n,\mathbf{L})\; a_p''(n,\mathbf{L})\; d\mathbf{L}. \end{aligned} \tag{A-11}$$

The superscript "a" distinguishes quantities characteristic of the adjoint system. The above sets of equations are instances of (1.3-2); each set is defined relative to the same basis fields and reference surfaces.

To find the relationships between transmitting and receiving characteristics, we let $\mathbf{E}'$, $\mathbf{H}'$

and $\mathbf{E}''$, $\mathbf{H}''$ be the fields corresponding to excitation of the respective systems by the incident waves represented by the following set of spectral variables:

$$a_0' = 1, \qquad a_p'(n, \mathbf{L}) \equiv 0,$$

$$a_0'' = 0, \qquad a_p''(n, \mathbf{L}) = \delta_{pq}\,\delta_{nm}\,\delta(\mathbf{L} - \mathbf{K}).$$

From (9) we find

$$-b_0''\eta_0 = b_q'(m, -\mathbf{K})\,\eta_m(K)$$

and from (10) and (11)

$$b_q'(m, \mathbf{K}) = S_{q0}(m, \mathbf{K}), \qquad b_0'' = S^a_{0q}(m, \mathbf{K}).$$

Hence [observing that $\eta_m(-K) = \eta_m(K)$] we obtain

$$\eta_0 S^a_{0q}(m, \mathbf{K}) = -\eta_m(K) S_{q0}(m, -\mathbf{K}). \tag{A-12a}$$

Similarly, by interchanging the patterns of excitation (or by interchanging the designations of the original and the adjoint system), we obtain

$$\eta_0 S_{0q}(m, \mathbf{K}) = -\eta_m(K) S^a_{q0}(m, -\mathbf{K}). \tag{A-12b}$$

To obtain the scattering reciprocity relations, we consider each system to be excited by an incident plane wave, as represented in the following scheme.

$$a_0' = 0, \qquad a_p'(n, \mathbf{L}) = \delta_{pp'}\delta_{nn'}\delta(\mathbf{L} - \mathbf{L}');$$

$$a_0'' = 0, \qquad a_p''(n, \mathbf{L}) = \delta_{pp''}\delta_{nn''}\delta(\mathbf{L} - \mathbf{L}'').$$

From (9) we find

$$b_{p''}'(n'', -\mathbf{L}'')\,\eta_{n''}(L'') = b_{p'}''(n', -\mathbf{L}')\,\eta_{n'}(L')$$

and from (10) and (11)

$$b_q'(m, \mathbf{K}) = S_{qp'}(m, \mathbf{K}; n', \mathbf{L}'), \qquad b_q''(m, \mathbf{K}) = S_{qp''}(m, \mathbf{K}; n'', \mathbf{L}'').$$

Hence (after changing variables to get rid of the primes), we obtain the set of scattering reciprocity relations

$$\eta_m(K) S_{pq}(m, \mathbf{K}; n, \mathbf{L}) = \eta_n(L) S^a_{qp}(n, -\mathbf{L}; m, -\mathbf{K}). \tag{A-13}$$

In this case interchanging S and S^a yields no further information.

If the reader has followed through any one of the above exercises, he will have no trouble in showing

$$S_{00} = S^a_{00}. \tag{A-14}$$

If we introduce the *reciprocity dyadic*

$$\boldsymbol{\eta} = \eta_1 \boldsymbol{\kappa}_1 \boldsymbol{\kappa}_1 + \eta_2 \boldsymbol{\kappa}_2 \boldsymbol{\kappa}_2 \tag{A-15}$$

and use the definitions (1.3–3), then (for example) (12b) becomes

$$\eta_0 \mathbf{S}_{0q}(\mathbf{K}) = \boldsymbol{\eta} \cdot \mathbf{S}^a_{q0}(-\mathbf{K}) \tag{A–16}$$

and (13) is summed up in

$$\boldsymbol{\eta}(\mathbf{K}) \cdot \mathbf{S}_{pq}(\mathbf{K}, \mathbf{L}) = \mathbf{S}^a_{qp}(-\mathbf{L}, -\mathbf{K})^T \cdot \boldsymbol{\eta}(\mathbf{L}), \tag{A–17}$$

where $\boldsymbol{\eta}(\mathbf{K})$ denotes $\boldsymbol{\eta}$ as a function of $\mathbf{K}$ and the superscript "T" denotes the transposed dyadic (obtained by transposing the elementary dyads involved). It is worth noting that the minus sign appearing between the members of (12b) does not appear in (16). The reciprocity dyadic is manifestly diagonal in the $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ basis; (4.2–3) shows what happens when $\boldsymbol{\eta}$ is presented in the $\mathbf{e}_x$, $\mathbf{e}_y$ basis.

The Lorentz reciprocity relation seems to have been the quite generally preferred basis for the derivation of transmission or coupling equations (similar, at least in function, to our (2–15)) [19, 21, 25, 26]. There is, however, no good reason why reciprocity should be invoked for that purpose: receiving characteristics can be defined analytically and operationally, independent of transmitting characteristics (as is done in this paper). Nevertheless, it is of some interest that the generalized Lorentz relation (8), or, more conveniently in the plane-wave framework, the lemma (9), can be used to derive transmission equations without invoking (ordinary) reciprocity.

5.2. Appendix B: Dissipative Characteristics of Media; Comparison for Mutually Adjoint Media

The concept that a medium may be "lossy," "lossless," or even "gainy" at a given point is familiar. We first need to give the mathematical expression of these qualitative properties of media. Then it will be easy to show that these properties are point-wise identical for mutually adjoint media.

According to properties of the complex Poynting's vector, dissipation or power loss per unit volume, $q(\mathbf{r})$, is given by

$$q(\mathbf{r}) = -\frac{1}{2} \operatorname{Re} \nabla \cdot (\mathbf{E} \times \overline{\mathbf{H}}). \tag{B–1}$$

In the medium described by (1.5–1), Maxwell's equations require

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega(\boldsymbol{\mu} \cdot \mathbf{H} + \boldsymbol{\nu} \cdot \mathbf{E}), \\ \nabla \times \mathbf{H} &= -i\omega(\boldsymbol{\epsilon} \cdot \mathbf{E} + \boldsymbol{\tau} \cdot \mathbf{H}). \end{aligned} \tag{B–2}$$

(Note that these equations are source-free in the sense that they are satisfied by $\mathbf{E} \equiv \mathbf{H} \equiv 0$.) Using a standard vector identity, we combine (1) and (2) to obtain

$$q(\mathbf{r}) = -\frac{1}{2} \operatorname{Re} \{i\omega[\overline{\mathbf{H}} \cdot \boldsymbol{\mu} \cdot \mathbf{H} - \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}} \cdot \overline{\mathbf{E}} + \overline{\mathbf{H}} \cdot (\boldsymbol{\nu} - \boldsymbol{\tau}^*) \cdot \mathbf{E}]\}, \tag{B–3}$$

where the superscript "*" denotes the Hermitian conjugate. For our purpose it is convenient to rearrange this expression to read

$$q(\mathbf{r}) = \frac{\omega}{4} [\overline{\mathbf{E}} - i\overline{\mathbf{H}}] \cdot \begin{pmatrix} i(\boldsymbol{\epsilon}^* - \boldsymbol{\epsilon}) & \boldsymbol{\nu}^* - \boldsymbol{\tau} \\ \boldsymbol{\nu} - \boldsymbol{\tau}^* & i(\boldsymbol{\mu}^* - \boldsymbol{\mu}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ i\mathbf{H} \end{pmatrix}. \tag{B–4}$$

The dissipative properties of the medium are thus seen to be determined by properties of the Hermitian matrix

$$\Lambda(\mathbf{r}) = \begin{pmatrix} i(\boldsymbol{\epsilon}^* - \boldsymbol{\epsilon}) & \boldsymbol{\nu}^* - \boldsymbol{\tau} \\ \boldsymbol{\nu} - \boldsymbol{\tau}^* & i(\boldsymbol{\mu}^* - \boldsymbol{\mu}) \end{pmatrix}. \tag{B-5}$$

which we call the loss matrix for the medium. For the present purpose the characteristics of the medium *at a given point* are appropriately classified according to the values assumable by $q(\mathbf{r})$ as $\mathbf{E}$ and $\mathbf{H}$, considered independent and arbitrary, vary. All possibilities are listed and named in table B–1.

There is no well-established terminology precisely fitting the physical properties being discussed. Terms generally chosen, and indeed the term "loss matrix," are biased by the tacit assumption that a medium normally is lossy. Thus if the loss matrix happens to be the zero matrix, we would ordinarily say that the medium is lossless at the point in question. But the medium would also be gainless at the same point. Neither physical nor technological restrictions prevent the occurrence of negative energy loss, which might be called gain (cf. Brand [51]).

Mathematical criteria for deciding the value class of Hermitian matrices are given by Mirsky [52]. In the lossless case, which is physically important but not counted as defining a value class by Mirsky, the loss matrix must be the zero matrix; this in turn requires $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^*$, $\boldsymbol{\tau} = \boldsymbol{\nu}^*$, and $\boldsymbol{\mu} = \boldsymbol{\mu}^*$.

TABLE B–1. *Classification of dissipative characteristics*

Values assumed by $q(\mathbf{r})$, $\mathbf{r}$ fixed	Value class of loss matrix at point $\mathbf{r}$	Dissipative characteristics of medium at point $\mathbf{r}$	
> 0	Positive definite	Unconditionally	lossy or passive
$\geqslant 0$	Positive semidefinite	Conditionally	
$\equiv 0$	Zero	Lossless, gainless or neutral	
$\leqslant 0$	Negative semidefinite	Conditionally	gainy or active
< 0	Negative definite	Unconditionally	
$\gtrless 0$	Indefinite	Indefinite	

Note: In the semidefinite cases both the zero and the nonzero values are to be assumable, and in the indefinite case both positive and negative values are to be assumable.

The medium adjoint to that described by (1.5–1) is defined and described by (1.5–2), according to which the adjoint expressions can be obtained by the replacements $\boldsymbol{\epsilon} \to \tilde{\boldsymbol{\epsilon}}$, $\boldsymbol{\tau} \to -\tilde{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \to -\tilde{\boldsymbol{\tau}}$, and $\boldsymbol{\mu} \to \tilde{\boldsymbol{\mu}}$. For the loss matrix this yields

$$\Lambda^a(\mathbf{r}) = \tilde{\Lambda}(\mathbf{r}) \tag{B-6}$$

i.e., $\Lambda^a(\mathbf{r})$ is the transpose of $\Lambda(\mathbf{r})$. This relation is sufficient to insure that the corresponding loss functions, $q(\mathbf{r})$ and $q^a(\mathbf{r})$, belong to the same value class (this can be seen, e.g., from the criteria given in Mirsky). We may say that the dissipative characteristics of a medium and its adjoint are the same, point by point. Whatever distribution of characteristics, including regions of active media (as in some antennas), is realized in one system will be realized in the adjoint system.

5.3. Appendix C: Two-Dimensional, Spatial Sampling or Interpolation Theorem

A variety of two-dimensional sampling theorems may be found in the literature [42, 53]. The methods of derivation often used are unnecessarily complicated for our purposes. For the

convenience of the reader we sketch a simple derivation of the simple desired result.

With the abbreviation $\mathbf{S}'_{02}(\mathbf{K}) \cdot \mathbf{S}_{10}(\mathbf{K}) \exp(i\gamma d) = f(\mathbf{K})$, the equation under discussion, (3.1–1), becomes

$$b'_0(\mathbf{P}) = a_0 F' \int f(\mathbf{K})\, e^{i\mathbf{K}\cdot\mathbf{P}} d\mathbf{K}. \tag{C–1}$$

We have already noted that the mathematical requirement that $f(\mathbf{K})$ be "band limited" may be fulfilled extraordinarily and almost arbitrarily well. Let us therefore assume that nonzero values of $f(\mathbf{K})$ occur only in a finite region K of wavenumber space. For simplicity, we take K to be rectangular, bounded by the lines $k_x = \pm k_1$, $k_y = \pm k_2$. Then (1) may be written

$$b'_0(\mathbf{P}) = a_0 F' \int_{-k_2}^{k_2} \int_{-k_1}^{k_1} f(\mathbf{K})\, e^{i\mathbf{K}\cdot\mathbf{P}} dk_x dk_y. \tag{C–2}$$

Further, $f(\mathbf{K})$ presumptively can be represented in K by a double Fourier series, with periods $2k_1$ and $2k_2$:

$$f(\mathbf{K}) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} C_{rs}\, e^{-i\mathbf{K}\cdot\mathbf{P}_{rs}}, \tag{C–3}$$

where the coefficients are given by the usual formula

$$C_{rs} = \frac{1}{4k_1 k_2} \int_{-k_2}^{k_2} \int_{-k_1}^{k_1} f(\mathbf{K})\, e^{i\mathbf{K}\cdot\mathbf{P}_{rs}}\, dk_x dk_y \tag{C–4}$$

with

$$\mathbf{P}_{rs} = \frac{r\pi}{k_1}\mathbf{e}_x + \frac{s\pi}{k_2}\mathbf{e}_y. \tag{C–5}$$

Comparing (4) and (2), we see that the C_{rs} are proportional to $b'_0(\mathbf{P})$ evaluated for $\mathbf{P} = \mathbf{P}_{rs}$, so that (3) may be written

$$f(\mathbf{K}) = \frac{1}{4k_1 k_2 a_0 F'} \sum_{r,s} b'_0(\mathbf{P}_{rs}) e^{-i\mathbf{K}\cdot\mathbf{P}_{rs}} \tag{C–6}$$

Thus $f(\mathbf{K})$ is completely determined by the sampling of data at the discrete points $\mathbf{P}_{rs}$. This is the result used in (3.1–4).

The usual objective in sampling theory is the reconstruction of the sampled function. In the present instance this is easily accomplished by substituting (6) in (2) and integrating. The result is an expression involving cardinal functions and expressing $b'_0(\mathbf{P})$ at all points in terms of its values at the lattice points.

One should observe the reciprocal relation between the size of the region K in wavenumber space and the size of the elementary cell in x, y space. The smaller the former, the larger the latter—and the smaller the required density of sampling points in x, y space.

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Optimal Facility Location for Normally and Exponentially Distributed Points *

I. N. Katz ** and Leon Cooper ***

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N destinations in the plane $\{P_j : j = 1, \ldots, N\}$ are given as independent random variables with specified probability densities, and the problem is to find the location of the point P which minimizes the expected sum of the Euclidean distances PP_j.

In this paper, upper bounds for the minimizing sum of distances are found in terms of solutions to corresponding deterministic problems and the first and second moments of the probability densities. Three commonly occurring classes of bivariate probability densities: (A) normal, (B) exponential, and (C) symmetric exponential, are then considered. Numerical tests are presented which show that in all cases, Steffensen's iteration is effective in accelerating convergence. Finally it is shown that in contrast to the deterministic case, P need not be in the convex hull of the means of P_j and a sufficient condition is given for P to be in this convex hull.

Key words: Always convergent algorithms; exponentially distributed points; facility location; location theory; normally distributed points; numerical analysis; optimization; stochastic Weber problem.

1. Introduction

A probabilistic extension of the classical Weber (Steiner) problem was introduced in [3][1] and [6]. N destinations in the plane, $\{P_j : j = 1, \ldots, N\}$, are given as random variables with specified probability density functions. It is desired to find the location of a point P with Cartesian coordinates $\bar{\alpha} = (\alpha, \beta)$ which minimizes the expected sum of the Euclidean distances $\overline{PP}_j$. A possible application of this problem to minimizing capital costs in water quality management is given in [6]. For simplicity we restrict ourselves in what follows to E^2.

Specifically, let P_j be a random variable with probability density function $\phi_j(\bar{x}_j)$, $\bar{x}_j = (x_j, y_j)$, $j = 1, \ldots, N$, where $\bar{x}_j \in S_j \subseteq E^2$. S_j is the sample space of P_j. Denote by $|\bar{x} - \bar{x}_j| = [(x - x_j)^2 + (y - y_j)^2]^{1/2}$ the Euclidean distance between $\bar{x} = (x, y)$ and $\bar{x}_j$. The expected value of $\overline{PP}_j$ is then $E(|\bar{x} - \bar{x}_j|) = \int\int_{S_j} |\bar{x} - \bar{x}_j| \phi_j(\bar{x}_j)\, dx_j\, dy_j$ and we wish to minimize the objective function

$$\psi(\bar{x}) = \sum_{j=1}^{N} \beta_j \int\int_{S_j} |\bar{x} - \bar{x}_j| \phi_j(\bar{x}_j)\, dx_j\, dy_j \tag{1}$$

over $\bar{x}$, where the β_j are given, positive weights.

In this paper $S_j = S$ for $j = 1, \ldots, N$; that is, all P_j are in the same sample space. We also restrict P to S. It was shown in [6] under mild assumptions on ϕ_j that $\psi(\bar{x})$ is strictly convex, and that the minimizing point P is unique. An iterative algorithm for finding P was obtained formally by setting $\nabla\psi(\bar{\alpha}) = 0$, by "solving" this equation in the form $\bar{\alpha} = H(\bar{\alpha})$, and then by passing to the iteration scheme $\bar{x}^{n+1} = \bar{H}(\bar{x}^n)$ which this suggests. Specifically,

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** Present address: Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130.
*** Present address: Department of Industrial Engineering and Operations Research, Southern Methodist University, Dallas, Texas 75275.
[1] Figures in brackets indicate the literature references at the end of this paper.

$$\bar{x}^{n+1}=\frac{\iint_S \frac{\bar{\xi}G(\bar{\xi})}{|\bar{x}^n-\bar{\xi}|}\,d\xi\,d\eta}{\iint_S \frac{G(\bar{\xi})}{|\bar{x}^n-\bar{\xi}|}\,d\xi\,d\eta}\equiv \bar{H}(\bar{x}^n) \qquad n=0,\,1,\,2,\,\ldots \tag{2}$$

where $\bar{\xi}=(\xi,\,\eta)$ and $G(\bar{\xi})=\sum_j \beta_j\phi_j(\bar{\xi})$. The algorithm was shown in [6] to be a descent method which is globally convergent to P and convergence was shown to be linear, locally. Although the considerations in [6] are for $S=E^2$, the same proofs apply with minor variations when $S=[0,\,\infty)\times[0,\,\infty)$.

In this paper, we first obtain upper bounds for

$$\bar{\psi}=\min_{\bar{x}\in S}\psi(\bar{x}) \tag{3}$$

in terms of the solutions to corresponding deterministic problems with the new destinations at the means of the random variables P_j. It is of interest to note that these bounds involve only the first and second moments of $\phi_j(\bar{x}_j)$. We also give a sufficient condition for P to be contained in an associated convex hull. This is important because whereas in the classical deterministic problem P is *always* in the convex hull of the destinations (see [5] and [7]), a corresponding statement for the probabilistic problem is not necessarily true, as we demonstrate later with a counterexample.

Next we consider three specific classes of probability density functions:

(A) $S=E^2$ and $\phi_j(\bar{x}_j)=$

$$\left(2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2}\right)^{-1}\exp\left\{-\frac{1}{2\sqrt{1-\rho_j^2}}\left[\left(\frac{x_j-\mu_{x_j}}{\sigma_{x_j}}\right)^2-2\rho_j\left(\frac{x_j-\mu_{x_j}}{\sigma_{x_j}}\right)\left(\frac{y_j-\mu_{y_j}}{\sigma_{y_j}}\right)+\left(\frac{y_j-\mu_{y_j}}{\sigma_{y_j}}\right)^2\right]\right\},$$

the bivariate normal density function,

(B) $S=[0,\,\infty)\times[0,\,\infty)$, and

$$\phi_j(\bar{x}_j)=\lambda_{x_j}\lambda_{y_j}\exp\{-\lambda_{x_j}x_j-\lambda_{y_j}y_j\},$$

the bivariate exponential density function, and

(C) $S=E^2$ and

$$\phi_j(\bar{x}_j)=\frac{1}{4}\lambda_{x_j}\lambda_{y_j}\exp\{-\lambda_{x_j}|x_j-\theta_{x_j}|-\lambda_{y_j}|y_j-\theta_{y_j}|\},$$

a bivariate exponential density function with mean at $(\theta_{x_j},\,\theta_{y_j})$ and standard deviation $(\sqrt{2}/\lambda_{x_j},\,\sqrt{2}/\lambda_{y_j})$. In each case we seek the minimizing $\bar{x}\in S$.

A major part of this paper deals with a special case of (A), namely the symmetric bivariate normal density when x_j and y_j are independent ($\rho_j=0$), and $\sigma_{x_j}=\sigma_{y_j}=\sigma_j$, $j=1,\,\ldots,\,N$. In this case, which is of considerable practical importance, *all* integrals appearing in (1) and (2) can be evaluated in closed form in terms of products of exponentials and modified Bessel Functions of orders 0 and 1. This may be very useful in practical computations, since tables or subroutines can

then be used instead of time-consuming numerical quadrature. We also prove that in this case P is in the convex hull of the means of the destinations $\bar{\mu}_j = (\mu_{x_j}, \mu_{y_j})$, which we denote by $\mathrm{co}\{\bar{\mu}_j\}$. By means of a simple counterexample with $N=2$ we show that even if $\rho_j = 0$, when $\sigma_{x_j} \neq \sigma_{y_j}$ P need *not* be in $\mathrm{co}\{\mu_j\}$. In the general case in (A) where the density is asymmetric, all double integrals are reducible to single integrals. This is also true for (B) and (C).

Finally, it was suggested in [6], that since convergence is locally linear in (2), Steffensen's iteration [4] for acceleration of convergence should be effective. We report the results of randomly generated numerical examples which confirm that this is indeed the case. We also compare our upper bound for $\bar{\psi}$ with its true value for some numerical examples.

2. Upper Bounds for $\bar{\psi}$

An upper bound for (3) follows from Schwarz's inequality. Since $\phi_j(\bar{x}_j) \geq 0$, and

$$\iint_S \phi_j(\bar{x}_j)\,dx_j\,dy_j = 1,$$

we have

$$\begin{aligned}
\psi(x, y) &= \sum_j \beta_j \iint_S \sqrt{(x-\xi)^2 + (y-\eta)^2}\,\phi_j(\xi, \eta)\,d\xi\,d\eta \\
&\leq \sum_j \beta_j \left[\iint_S [(x-\xi)^2 + (y-\eta)^2]\phi_j(\xi, \eta)\,d\xi\,d\eta \right]^{1/2} \left[\iint_S \phi_j(\xi, \eta)\,d\xi\,d\eta \right]^{1/2} \\
&= \sum_j \beta_j \left[x^2 - 2x\mu_{x_j} + E[x_j^2] + y^2 - 2y\mu_{y_j} + E[y_j^2] \right]^{1/2} \\
&= \sum_j \beta_j \left[(x-\mu_{x_j})^2 + (y-\mu_{y_j})^2 + \sigma_{x_j}^2 + \sigma_{y_j}^2 \right]^{1/2}
\end{aligned}$$

where $\mu_{x_j} = E[x_j]$, $\mu_{y_j} = E[y_j]$ are the respective x and y means of $\phi_j(\bar{x}_j)$, and where $\sigma_{x_j}^2 = E[x_j^2] - \mu_{x_j}^2$, $\sigma_{y_j}^2 = E[y_j^2] - \bar{\mu}_{y_j}^2$ are the respective x and y variances of $\phi_j(\bar{x}_j)$. This gives

$$\bar{\psi} \leq \min_{x \in S} \sum_j \beta_j [(x-\mu_{x_j})^2 + (y-\mu_{y_j})^2 + \sigma_j^2]^{1/2} \tag{4}$$

where $\sigma_j^2 = \sigma_{x_j}^2 + \sigma_{y_j}^2$. A coarser but more easily calculated upper bound follows immediately from (4):

$$\begin{aligned}
\bar{\psi} &\leq \min_{x \in S} \sum_j \beta_j \left([(x-\mu_{x_j})^2 + (y-\mu_{y_j})^2]^{1/2} + \sigma_j \right) \\
&= \sum_j \beta_j \sigma_j + \min_{x \in S} \sum_j \beta_j [(x-\mu_{x_j})^2 + (y-\mu_{y_j})^2]^{1/2} = \bar{\psi}\ \text{Bound.}
\end{aligned} \tag{5}$$

The second term on the right in (5) is the minimum weighted sum of the distances in a deterministic problem with the destinations at the means of P_j, and the first term is a weighted sum of the standard deviations of the P_j. Note that for any $\phi_j(\bar{x}_j)$, the bounds in (4) and (5) depend only upon its first and second moments.

3. The Bivariate Normal Density Function

Assume that $\phi_j(\bar{x}_j)$ for $j = 1, \ldots, N$ is given by (A). We now show that

$$\text{if } \rho_j = 0 \text{ and } \sigma_{x_j} = \sigma_{y_j} = \sigma_j \text{ for } j = 1, \ldots, N \text{ then } \bar{\alpha} \in co\{\bar{\mu}_j\}_{j=1}^{N} \tag{6}$$

In this case

$$\phi_j(\bar{x}_j) = (2\pi\sigma_j^2)^{-1} \exp\left\{-\frac{1}{2\sigma_j^2}\left[(x_j - \mu_{x_j})^2 + (y_j - \mu_{y_j})^2\right]\right\}. \tag{7}$$

We first prove two lemmas which are useful in locating $\bar{\alpha}$.

LEMMA 1: *Suppose $z_i(\bar{y})$ are differentiable, real valued functions defined on E^2, $i = 1, \ldots, N$ and let $\bar{\gamma}$ be a point at which $z(\bar{y}) = \Sigma\, z_i(\bar{y})$ is minimized. Then $\bar{\gamma} \in co\{\bar{y}_i\}_{i=1}^{N}$ is assured under the following two further assumptions:*

(a) *The level curves $S_i = \{\bar{y} \in E^2 \mid z_i(\bar{y}) = c\}$ are concentric circles centered at $\bar{y}_i$, that is, $S_i = \{\bar{y} \mid |\bar{y} - \bar{y}_i| = r(c)\}$.*
(b) *For some k, $\nabla z_k(y) \neq 0$ if $\bar{y} \neq \bar{y}_k$.*

PROOF: Assume, to the contrary, that $\bar{\gamma} \notin co\{\bar{y}_i\}_{i=1}^{N}$. Then there is a line l containing $\bar{\gamma}$, with a normal $\bar{a}$, such that all $\bar{y}_i$, $i = 1, \ldots, N$ are contained in an open half plane bounded by l. For $\bar{y}$ on l, we have $\bar{a} \cdot \bar{y} = \bar{a} \cdot \bar{\gamma} = b$, and $\bar{a} \cdot (\bar{y}_i - \bar{\gamma}) > 0$, for $i = 1, \ldots, N$. Now, because of (a),

$$\nabla z_i(\bar{\gamma}) = |\nabla z_i(\gamma)| \frac{\bar{\gamma} - \bar{y}_i}{|\bar{\gamma} - \bar{y}_i|}$$

and because of (b) $|\nabla z_k(\bar{\gamma})| > 0$. Therefore, we have

$$(\nabla z(\bar{\gamma})) \cdot \bar{a} = \sum_i |\nabla z_i(\bar{\gamma})| \frac{\bar{\gamma} - \bar{y}_i}{|\bar{\gamma} - \bar{y}_i|} \cdot \bar{a} < 0$$

which implies that $z(\bar{y})$ is decreasing in the direction $\bar{a}$ from $\bar{\gamma}$. This contradicts the assumption that $z(\bar{y})$ is minimized at $\bar{\gamma}$, hence $\bar{\gamma} \in co\{\bar{y}_i\}_{i=1}^{N}$.

LEMMA 2: *Suppose that the probability density function $\phi_j(\bar{x}_j)$ is radially symmetric about its mean μ_j; that is, suppose $\phi_j(\bar{x}_j) = f_j(|\bar{x}_j - \bar{\mu}_j|)$, $\bar{x}_j \in E^2$. Then*

$$z_j(\bar{x}) = \beta_j \int\int_{E_2} |\bar{x} - \bar{\xi}|\, \phi_j(\bar{\xi}) d\xi d\eta \tag{8}$$

satisfies assumption (a) in lemma 1, with $\bar{y}_j = \bar{\mu}_j$.

PROOF: Use polar coordinates $\xi - \mu_{x_j} = \rho \cos\Theta$, $\eta - \mu_{y_j} = \rho \sin\Theta$ in (8) to obtain

$$\begin{aligned} z_j(\bar{x}) &= \beta_j \int\int_{E^2} |\bar{x} - \bar{\xi}|\, \phi_j(\bar{\xi})\, d\xi d\eta = \beta_j \int\int_{E^2} |\bar{x} - \bar{\xi}|\, f_j(|\bar{\xi} - \bar{\mu}_j|)\, d\xi d\eta \\ &= \int_0^{\infty}\int_0^{2\pi} \rho\left[(x - \mu_{x_j} - \rho\cos\Theta)^2 + (y - \mu_{y_j} - \rho\sin\Theta)^2\right]^{1/2} f_j(\rho)\, d\Theta d\rho \\ &= \int_0^{\infty} \rho f_j(\rho) d\rho \int_0^{2\pi} \left[r_j^2 + \rho^2 - 2\rho r_j \cos(\Theta - \Phi_j)\right]^{1/2} d\Theta \end{aligned} \tag{9}$$

where $r_j^2 = (x - \mu_{x_j})^2 + (y - \mu_{y_j})^2$,

$$\cos \Phi_j = (x - \mu_{x_j})/r_j, \qquad \sin \Phi_j = (y - \mu_{y_j})/r_j. \tag{10}$$

Using the periodicity of the integrand over $[0, 2\pi]$, and letting $\Psi = \Theta - \Phi_j$, (9) becomes

$$z_j(\bar{x}) = \int_0^\infty \rho f_j(\rho) d\rho \int_0^{2\pi} [r_j^2 + \rho^2 - 2\rho r_j \cos \Psi]^{1/2} d\Psi \tag{11}$$

which implies that $z_j(\bar{x})$ in its dependence upon $\bar{x}$ is a function of r_j, The Euclidean distance to $\bar{\mu}_j$, alone. Hence, the level curves of $z_j(\bar{x})$ are concentric circles centered about $\bar{\mu}_j$. This proves the lemma.

Now we wish to apply lemma 1 with

$$z_j(\bar{x}) = \beta_j \int\int_{E^2} | \bar{x} - \bar{\xi} | \, \phi_j(\bar{\xi}) d\xi d\eta$$

and with $\phi_j(\bar{\xi})$ given by (7). It follows from [5] with $N = 1$ that $z_j(\bar{x})$ is strictly convex on E^2. Also

$$\nabla z_j(\bar{x}) = \beta_j \int\int_{E^2} \frac{\bar{x} - \bar{\xi}}{|\bar{x} - \bar{\xi}|} \phi_j(\bar{\xi}) \, d\xi d\eta,$$

and an easy computation shows that $\nabla z_j(\bar{\mu}_j) = 0$ where $\bar{\mu}_j = (\mu_{x_j}, \mu_{y_j})$. Since $z_j(\bar{x})$ is strictly convex, $\nabla z_j(\bar{x}) \neq 0$ if $\bar{x} \neq \bar{\mu}_j$ for *all* $j = 1, \ldots, N$. Hence $z_j(\bar{x})$ satisfies assumption (b) of lemma 1. In order to show that $z_j(\bar{x})$ satisfies assumption (a) of lemma 1, observe that from (7) we have

$$\phi_j(\bar{x}_j) = (2\pi\sigma_j^2)^{-1} \exp\left\{ \frac{-1}{2\sigma_j^2} |\bar{x}_j - \bar{\mu}_j|^2 \right\}$$

so that $\phi_j(\bar{x}_j) = f_j(|\bar{x}_j - \bar{\mu}_j|)$. By lemma 2, then, $z_j(\bar{x})$ satisfies condition (a) of lemma 1, with $\bar{y}_j = \bar{\mu}_j$. Now apply lemma 1 with $\bar{y}_j = \bar{\mu}_j$ to obtain (6).

The result in (6) is not trivial. In the classical deterministic problem $\bar{\alpha}$ is always contained in $\text{co}\{P_j\}_{j=1}^N$ where P_j are the given destinations (see [5] and [7]). However, in the probabilistic case, we now show by means of a simple counterexample with $N = 2$, that even when the ϕ_j are normal bivariate density functions with $\rho_j = 0$, if $\sigma_{x_j} \neq \sigma_{y_j}$ then $\bar{\alpha}$ is not necessarily in $\text{co}\{\bar{\mu}_j\}_{j=1}^N$.

Consider the problem: $N = 2$, $\phi_j(\bar{x}_j)$ for $j = 1, 2$ given by (A),

$$\beta_1 = \beta_2 = 1, \mu_{x_1} = 1, \mu_{y_1} = 0, \sigma_{x_1} = \sigma, \sigma_{y_1} = 1; \mu_{x_2} = 0, \mu_{y_2} = 1, \sigma_{x_2} = 1, \sigma_{y_2} = \sigma;$$

$\rho_j = 0$. In this problem $\text{co}\{\bar{\mu}_j\}_{j=1}^2 = \{(x, y) | x + y = 1, 0 \leq x \leq 1\}$, the closed line segment joining the means. From symmetry considerations, it is easy to show that for each fixed σ, $0 < \sigma < \infty$, the minimizing point $\bar{\alpha} = (\alpha, \beta)$ satisfies $\alpha = \beta$. Therefore $\bar{\alpha}$ is in $\text{co}\{\bar{\mu}_j\}_{j=1}^2$ if and only if $\alpha = \beta = \frac{1}{2}$. It also follows using $\nabla \psi(\bar{\alpha}) = 0$ that $\bar{\alpha}$ satisfies $\frac{\partial \psi}{\partial x}(\alpha, \alpha) = 0$ for each σ:

$$\frac{\partial \psi}{\partial x}(\alpha, \alpha) = (2\pi\sigma)^{-1} \left\{ \iint_{E^2} \frac{\alpha - \xi}{|\bar{\alpha} - \bar{\xi}|} \exp\left\{ -\frac{1}{2} \left(\xi^2 + \frac{(\eta - 1)^2}{\sigma^2} \right) \right\} d\xi d\eta \right.$$

$$\left. + \iint_{E^2} \frac{\alpha - \xi}{\alpha - \xi} \exp\left\{ -\frac{1}{2} \left(\frac{(\xi - 1)^2}{\sigma^2} + \eta^2 \right) \right\} d\xi d\eta \right\} = 0. \tag{12}$$

We give the following argument: As $\sigma \to 0$ it is well known (see [2], p. 324 and [8], chapter 1, for example) that in the sense of distributions $(\sqrt{2\pi}\,\sigma)^{-1} \exp\{-1/2\,(u-1)^2/\sigma^2\} \to \delta(u-1)$, the Dirac-delta function centered at $u=1$. In each integral of (12), one dimension of the probabilistic problem then becomes deterministic with the corresponding coordinate of the destination fixed at 1. Letting $\sigma \to 0$ in (12) we obtain

$$f(\alpha) = \int_{-\infty}^{\infty} \frac{\alpha - \xi}{\sqrt{(\alpha-\xi)^2 + (\alpha-1)^2}} e^{-1/2\xi^2}\, d\xi + \int_{-\infty}^{\infty} \frac{\alpha - 1}{\sqrt{(\alpha-1)^2 + (\alpha-\eta)^2}}\, e^{-1/2\eta^2} d\eta$$

$$= \int_{-\infty}^{\infty} \frac{2\alpha - \xi - 1}{\sqrt{(\alpha-\xi)^2 + (\alpha-1)^2}}\; e^{-1/2\xi^2} d\xi = 0. \tag{13}$$

We now show that the unique α satisfying (13), also satisfies $1/2 < \alpha < 1$. Continuity arguments then imply that for σ sufficiently small the unique α satisfying (12) is also such that $1/2 < \alpha < 1$, i.e., for σ sufficiently small $\bar{\alpha} \notin \text{co}\{\bar{\mu}_j\}_{j=1}^2$, thereby providing a counterexample. We have

$$f\left(\frac{1}{2}\right) = -\int_{-\infty}^{\infty} \frac{\xi}{\sqrt{\left(\xi - \frac{1}{2}\right)^2 + \frac{1}{4}}}\; e^{-1/2\xi^2} d\xi$$

$$= -\int_0^{\infty} \xi \left(\frac{1}{\sqrt{\left(\xi - \frac{1}{2}\right)^2 + \frac{1}{4}}} - \frac{1}{\sqrt{\left(\xi + \frac{1}{2}\right)^2 + \frac{1}{4}}} \right) e^{-1/2\xi^2}\; d\xi < 0$$

and

$$f(1) = \int_{-\infty}^{\infty} \frac{1-\xi}{|1-\xi|}\, e^{-\frac{1}{2}\xi^2}\, d\xi = \int_{-1}^{1} e^{-\frac{1}{2}\xi^2}\, d\xi > 0.$$

Hence α satisfying (12) is such that $\frac{1}{2} < \alpha < 1$, which concludes the proof. (In fact $\alpha \to 0.66$ approximately, as $\sigma \to 0$, as seen from table 1.)

For purposes of illustration we have solved (12) numerically using an iterative scheme analogous to (2). In table 1 we give α for various values of σ, $0 < \sigma < \infty$. As σ goes from 0 to ∞, α goes monotonically from 0.660 to 0. Therefore for *no* value of σ except $\sigma = 1$ (when $\sigma_{x_1} = \sigma_{y_1} = 1 = \sigma_{x_2} = \sigma_{y_2}$) is $\bar{\alpha}$ in $\text{co}\{\bar{\mu}_j\}_{j=1}^2$. This is in marked contrast to the deterministic case when $\bar{\alpha}$ is *always* in the convex hull of the destinations. We also give in table 1, $\bar{\psi}$, the bound for $\bar{\psi}$ from equations (4), ψ BOUND from (5), and M, the number of iterations of (2) to achieve three place accuracy. The bound for $\bar{\psi}$ from eq (4) was calculated from an obvious extension of the classical iteration for solving the Weber problem (see [5], or [6], for example).

TABLE 1. *Counterexample*

σ	α	M	$\bar{\psi}$	Bound for $\bar{\psi}$ from (4)	$\bar{\psi}$ Bound from (5)
1/32	0.660	7	2.123	2.450	3.415
1/16	.658	7	2.125	2.452	3.418
1/8	.650	7	2.134	2.462	3.430
1/4	.623	7	2.173	2.500	3.476
1/2	.567	8	2.326	2.646	3.650
1*	.500*	–	2.814*	–	4.243
2	.416	9	4.074	4.690	4.886
4	.391	11	6.969	8.367	9.660
8	.259	11	13.108	16.186	17.539
16	.122	12	25.708	32.094	33.477
32	.043	12	51.138	64.047	65.445
64	.012	12	102.136	128.023	129.430
128	.003	12	204.202	256.012	257.422
256	.001	12	408.370	512.006	513.418
512	.000	12	816.723	1024.003	1025.42

*Computed from closed form expressions (see (24)).
$\bar{\psi}$ BOUND $= \sqrt{2}+2\sqrt{1+\sigma^2}$.
The minimum $\psi(x, y)=\psi_{\min}$ in the corresponding deterministic problem with P_j at $\bar{\mu}_j$ is: $\psi_{\min}=\sqrt{2}$.

3.1. Evaluation of Integrals in the Symmetric Uncorrelated Case

We now give explicit closed form expressions for the evaluation of all of the integrals in (2) when $\rho_j=0$ and $\sigma_{x_j}=\sigma_{y_j}=\sigma_j$ for $j=1, \ldots, N$. In this case $\phi_j(\bar{x}_j)$ is given by (7) and the right-hand side of (2) becomes

$$\bar{H}(\bar{x})=\frac{\sum_j \beta_j \bar{N}_j(\bar{x})}{\sum \beta_j D_j(\bar{x})} \tag{14}$$

where

$$\bar{N}_j(\bar{x})=(2\pi\sigma_j^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\xi}[(x-\xi)^2+(y-\eta)^2]^{-1/2}\exp\left\{-\frac{1}{2\sigma_j}[(\xi-\mu_{x_j})^2+(\eta-\mu_{y_j})^2]\right\}d\xi d\eta \tag{15}$$

$$D_j(\bar{x})=(2\pi\sigma_j^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(x-\xi)^2+(y-\eta)^2]^{-1/2}\exp\left\{-\frac{1}{2\sigma_j}[(\xi-\mu_{x_j})^2+(\eta-\mu_{y_j})^2]\right\}d\xi d\eta. \tag{16}$$

We evaluate the first component of $\bar{N}_j(\bar{x})$, which for simplicity we denote by $N_j(\bar{x})$. The evaluation of the second component follows similarly. Substitute $\xi-x=\rho\cos\theta$, $\eta-y=\rho\sin\theta$ to obtain

$$N_j(\bar{x})=(2\pi\sigma_j^2)^{-1}\int_0^{\infty}\int_0^{2\pi}(x+\rho\cos\theta)\exp\left\{-\frac{1}{2\sigma_j^2}[(x-\mu_{x_j}+\rho\cos\theta)^2 + (y-\mu_{y_j}+\rho\sin\theta)^2]\right\}d\theta d\rho = K_j(\bar{x})+L_j(\bar{x}). \tag{17}$$

Use the notation in (10) and let

$$J_j(\rho)=\int_0^{2\pi} \exp\{-(\rho r_j/\sigma_j^2)\cos(\theta-\Phi_j)\}d\theta$$

$$=\int_0^{2\pi} \exp\{-(\rho r_j/\sigma_j^2)\cos\Psi\}d\Psi$$

where we have substituted $\theta-\Phi_j=\Psi$ and used the periodicity of the integrand. Then, we have

$$K_j(\bar{x})=x(2\pi\sigma_j^2)^{-1}\exp\{-r_j^2/2\sigma_j^2\}\int_0^{\infty}\exp\{-(\rho^2/2\sigma_j^2)\}J_j(\rho)d\rho.$$

Reference to [1], p. 376, 9.6.16 or 9.6.19 shows that

$$J_j(\rho)=2\pi I_0(\rho r_j/\sigma_j^2)$$

where I_k is the modified Bessel function of the first kind of order k. Reference to [1], p. 487, 11.4.31 then gives

$$K_j(\bar{x})=(x/\sigma_j)(\pi/2)^{1/2}\exp\{-r_j^2/4\sigma_j^2\}I_0\{r_j^2/4\sigma_j^2\}. \tag{18}$$

In order to evaluate $L_j(\bar{x})$ note that

$$L_j(\bar{x})=(2\pi\sigma_j^2)^{-1}\exp\{-r_j^2/2\sigma_j^2\}\int_0^{\infty}\rho\exp\{-\rho^2/2\sigma_j^2\}M_j(\rho)d\rho \tag{19}$$

where

$$M_j(\rho)=\int_0^{2\pi}\cos\theta\exp\{(-\rho r_j/\sigma_j^2)\cos(\theta-\Phi_j)\}d\theta.$$

Let $\theta-\Phi_j=\Psi$, use the peridoicity of the integrand and refer to [1], p. 376, 9.6.19, to obtain

$$M_j(\rho)=-2\pi((x-\mu_{x_j})/r_j)I_1\{\rho r_j/\sigma_j^2\}.$$

Substituting in (19), this gives

$$L_j(\rho)=-((x-\mu_{x_j})/r_j)\exp\{-r_j^2/2\sigma_j^2\}\left[\sigma_j^{-2}\int_0^{\infty}\rho\exp\{-\rho^2/2\sigma_j^2\}I_1\{\rho r_j/\sigma_j^2\}d\rho\right].$$

Integrating by parts in the last integral, we obtain

$$L_j(\rho)=-((x-\mu_{x_j})/r_j)\exp\{-r_j^2/2\sigma_j^2\}.$$

$$\left[-I_1\{\rho r_j/\sigma_j^2\}\exp\left\{-\rho^2/2\sigma_j^2\right|_{\rho=0}^{\rho=\infty}+(r_j/\sigma_j^2)\int_0^{\infty}\exp\{-\rho^2/2\sigma_j^2\}I_1'\{\rho r_j/\sigma_j^2\}d\rho\right]. \tag{20}$$

The integrated term vanishes at $\rho=0$ since $I_1(0)=0$. At $\rho=\infty$ we use $I_1(z)\sim e^z(2\pi z)^{-1/2}$ (see [1], p. 377, 9.7.1) to show that the integrated term vanishes there too. The expression in (20) can be further simplified by using $2I_1'=I_0+I_2$ (see [1], p. 376, 9.6.26). This simplification combined with use of [1], p. 487, 11.4.31 gives

$$L_j(\rho) = -((x-\mu_{x_j})/2\sigma_j)(\pi/2)^{1/2} \exp\{-r_j^2/4\sigma_j^2\}(I_0\{r_j^2/4\sigma_j^2\} + I_1\{r_j^2/4\sigma_j^2\}). \tag{21}$$

Substituting (18) and (21) in (17) gives the explicit closed form expression for a typical double integral in (14),

$$N_j(\bar{x}) = (2\sigma_j)^{-1}(\pi/2)^{1/2} \exp\{-r_j^2/4\sigma_j^2\}((x+\mu_{x_j})I_0\{r_j^2/4\sigma_j^2\} - (x-\mu_{x_j})I_1\{r_j^2/4\sigma_j^2\})\}. \tag{22}$$

Calculations completely analogous to those which gave $K_j(\bar{x})$ in (18) also yield

$$D_j(\bar{x}) = \sigma_j^{-1}(\pi/2)^{1/2} \exp\{-r_j^2/4\sigma_j^2\} I_0\{r_j^2/4\sigma_j^2\}. \tag{23}$$

(22) and (23) are the closed form expressions required for the iterative algorithm (2), in this special case.

The equation for the minimizing $\bar{x}$, $\bar{x} = \bar{H}(\bar{x})$ can now be written as

$$\bar{x} = \frac{\frac{1}{2}\sum_j \beta_j \sigma_j^{-1} \exp\{-r_j^2/4\sigma_j^2\}((\bar{x}+\bar{\mu}_j)I_0\{r_j^2/4\sigma_j^2\} - (\bar{x}-\bar{\mu}_j)I_1\{r_j^2/4\sigma_j^2\})}{\sum_j \beta_j \sigma_j^{-1} \exp\{-r_j^2/4\sigma_j^2\} I_0\{r_j^2/4\sigma_j^2\}}. \tag{24}$$

It is of interest to note that (24) can be rewritten in the equivalent form

$$\bar{x} = \sum_{k=1}^{N} \lambda_k \bar{\mu}_k, \tag{25}$$

where

$$\lambda_k = \frac{\beta_k \sigma_k^{-1} \exp\{-r_k^2/4\sigma_k^2\}(I_0\{r_k^2/4\sigma_k^2\} + I_1\{r_k^2/4\sigma_k^2\})}{\sum_{j=1}^{N} \sigma_j^{-1} \exp\{-r_j^2/4\sigma_j^2\}(I_0\{r_j^2/4\sigma_j^2\} + I_1\{r_j^2/4\sigma_j^2\})}$$

satisfy $0 < \lambda_k < 1$, $\Sigma\lambda_k = 1$. Therefore (25) is a closed form expression (in terms of $\bar{x}$) for $\bar{x}$ as a convex combination of $\bar{\mu}_k$, with constants λ_k. We proved earlier that in the case of bivariate normal density functions with x_j and y_j independent and $\sigma_{x_j} = \sigma_{y_j} = \sigma_j$, such a combination always exists since $\bar{x} \in \text{co}\{\bar{\mu}_k\}_{k=1}^{N}$.

It is also possible to evaluate the objective function $\Psi(\bar{x})$ in (1) without numerical quadrature. We have

$$\psi(\bar{x}) = \sum_j \beta_j (2\pi\sigma_j^2)^{-1} \iint_{E^2} |\bar{x}-\bar{\xi}| \exp\left\{\frac{-1}{2\sigma_j^2}|\bar{\xi}-\bar{\mu}_j|^2\right\} d\xi d\eta$$

$$= \sum_j z_j(\bar{x})$$

and

$$z_j(\bar{x}) = \beta_j (2\pi\sigma_j^2)^{-1} \exp\{-r_j^2/2\sigma_j^2\} \int^{\infty} \rho^2 J_j(\rho) \exp\{-\rho^2/2\sigma_j^2\}\, d\rho$$

$$= \beta_j \sigma_j^{-2} \exp\{-r_j^2/2\sigma_j^2\} \int_0^{\infty} \rho^2 I_0(\rho r_j/\sigma^2) \exp\{-\rho^2/2\sigma^2\}\, d\rho. \tag{26}$$

Refer to [1] p. 487, 11.4.31 and differentiate with respect to the parameter a^2 appearing there. This gives a closed form expression for the integral in (26), and (26) then becomes

$$z_j(\bar{x}) = (\beta_j/\sigma_j)(\pi/2)^{1/2} \exp\{-r_j^2/4\sigma_j^2\}\,((r_j^2/2)(I_0(r_j^2/4\sigma_j^2) + I_1(r_j^2/4\sigma_j^2)) + \sigma_j^2 I_0(r_j^2/4\sigma_j^2)).$$

3.2. Evaluation of Integrals in the General Case

To conclude our analysis of the bivariate normal density function we now show that even in the general case (when the density is not symmetric and uncorrelated), all of the double integrals appearing in (1) and (2) can be evaluated as single integrals in terms of error functions. The iteration in (2) then becomes considerably more efficient because only single integrals need be evaluated by numerical quadrature, using subroutines available for the calculation of the error function.

In the general case $\phi_j(\bar{x}_j)$ is given by (A) and the right-hand side of (2) becomes

$$\bar{H}(\bar{x}) = \frac{\sum_j \beta_j \bar{N}_j^*(\bar{x})}{\sum_j \beta_j D_j^*(\bar{x})} \tag{27}$$

where

$$\bar{N}_j^*(\bar{x}) = (2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1}.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\xi}[(x-\xi)^2+(y-\eta)^2]^{-1/2} \exp\left\{-\frac{1}{2\sqrt{1-\rho_j^2}}\left[\left(\frac{\xi-\mu_{x_j}}{\sigma_{x_j}}\right)^2 - 2\rho_j\left(\frac{\xi-\mu_{x_j}}{\sigma_{x_j}}\right)\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)+\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)^2\right]\right\} d\xi\, d\eta \tag{28}$$

$$D_j^*(\bar{x}) = (2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1}.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x-\xi)^2+(y-\eta)^2]^{-1/2} \exp\left\{-\frac{1}{2\sqrt{1-\rho_j^2}}\left[\left(\frac{\xi-\mu_{x_j}}{\sigma_{x_j}}\right)^2 - 2\rho_j\left(\frac{\xi-\mu_{y_j}}{\sigma_{x_j}}\right)\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)+\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)^2\right]\right\} d\xi\, d\eta. \tag{29}$$

Again, we evaluate only the first component of $\bar{N}_j^*(\bar{x})$ which, for simplicity, we denote by $N_j^*(\bar{x})$. The evaluation of the second component follows similarly. Substitute, as before, $\xi - x = \rho\cos\theta$, $\eta - y = \rho\sin\theta$. The exponent in brackets in (28) and (29) then becomes

$$\left[\left(\frac{x-\mu_{x_j}+\rho\cos\theta}{\sigma_{x_j}}\right)^2 - 2\rho_j\left(\frac{x-\mu_{x_j}+\rho\cos\theta}{\sigma_{x_j}}\right)\left(\frac{y-\mu_{y_j}+\rho\sin\theta}{\sigma_{y_j}}\right)+\left(\frac{y-\mu_{y_j}+\rho\sin\theta}{\sigma_{y_j}}\right)^2\right]$$

$$= \rho^2 A_j + 2\rho B_j + r_j^* = A_j\left(\rho+\frac{B_j}{A_j}\right)^2 - \frac{B_j^2}{A_j} + r_j^*$$

where

$$A_j \equiv A_j(\theta) = \frac{\cos^2\theta}{\sigma_{x_j}^2} - \frac{2\rho_j \cos\theta \sin\theta}{\sigma_{x_j}\sigma_{y_j}} + \frac{\sin^2\theta}{\sigma_{y_j}^2}$$

$$B_j \equiv B_j(\theta) = \frac{(x-\mu_{x_j})\cos\theta}{\sigma_{x_j}^2} - \rho_j \frac{(x-\mu_{x_j})\sin\theta + (y-\mu_{y_j})\cos\theta}{\sigma_{x_j}\sigma_{y_j}} + \frac{(y-\mu_{y_j})\sin\theta}{\sigma_{y_j}^2}$$

$$r_j^* = \left(\frac{x-\mu_{x_j}}{\sigma_{x_j}}\right)^2 - 2\rho_j \frac{(x-\mu_{x_j})(y-\mu_{y_j})}{\sigma_{x_j}\sigma_{y_j}} + \left(\frac{y-\mu_{y_j}}{\sigma_{y_j}}\right)^2.$$

Note that $A_j \equiv A_j(\theta)$ is never zero for $0 < \sigma_{x_j}, \sigma_{y_j} < \infty$. $N_j^*(\bar{x})$ now becomes

$$N_j^*(\bar{x}) = (2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1} \exp\{-r_j^*/(2\sqrt{1-\rho_j^2})\} \int_0^{2\pi} \exp\{B_j^2/(2A_j\sqrt{1-\rho_j^2})\} P_j(\theta)\, d\theta$$

where

$$P_j(\theta) = \int_0^\infty (x+\rho\cos\theta) \exp\{-A_j(\rho + B_j/A_j)^2/(2\sqrt{1-\rho_j^2})\}\, d\rho.$$

We now show that $P_j(\theta)$ can be evaluated in closed form. For each fixed θ, let

$$t = \left(\rho + \frac{B_j}{A_j}\right)\left(\frac{A_j}{2\sqrt{1-\rho_j^2}}\right)^{1/2}; \tag{30}$$

then

$$P_j(\theta) = (2\sqrt{1-\rho_j^2}/A_j)^{1/2} \cdot$$

$$\int_{B_j/(2A_j\sqrt{1-\rho_j^2})^{1/2}}^\infty \left[\left(x - \frac{B_j}{A_j}\cos\theta\right) + (2\sqrt{1-\rho_j^2}/A_j)^{1/2}(\cos\theta)t\right] e^{-t^2} dt$$

$$= (\pi\sqrt{1-\rho_j^2}/2A_j)^{1/2}\left(x - \frac{B_j}{A_j}\cos\theta\right) \operatorname{erfc}\{B_j/(2A_j\sqrt{1-\rho_j^2})^{1/2}\}$$

$$+ (\sqrt{1-\rho_j^2}/A_j)(\cos\theta) \exp\{-B_j^2/(2A_j\sqrt{1-\rho_j^2})\}$$

where erfc is the complementary error function (defined, for instance, in [1], p. 297).

Similarly, for the terms in the denominator of (27), we have

$$D_j^*(\bar{x}) = (2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1} \exp\{-r_j^*/(2\sqrt{1-\rho_j^2})\} \int_0^{2\pi} \exp\{B_j^2/2A_j\sqrt{1-\rho^2}\} Q_j(\theta)\, d\theta$$

where

$$Q_j(\theta) = \int_0^\infty \exp\{-A_j(\rho + B_j/A_j)^2/(2\sqrt{1-\rho_j^2})\}\, d\rho$$

$$= (\pi\sqrt{1-\rho_j^2}/2A_j)^{1/2} \operatorname{erfc}\{B_j/(2A_j\sqrt{1-\rho_j^2})^{1/2}\}.$$

Aslo, to evaluate $\psi(\bar{x})$ in (1) we have

$$\psi(\bar{x}) = \sum_j z_j(\bar{x}) \tag{31}$$

where

$$z_j(\bar{x}) = \beta_j(2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(x-\xi)^2+(y-\eta)^2]^{1/2}\left\{-\frac{1}{2\sqrt{1-\rho_j^2}}\left[\left(\frac{\xi-\mu_{x_j}}{\sigma_{x_j}}\right)^2\right.\right.$$

$$\left.\left.-2\rho_j\left(\frac{\xi-\mu_{x_j}}{\sigma_{x_j}}\right)\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)+\left(\frac{\eta-\mu_{y_j}}{\sigma_{y_j}}\right)^2\right]\right\}d\xi d\eta.$$

This leads to

$$z_j(\bar{x}) = \beta_j(2\pi\sigma_{x_j}\sigma_{y_j}\sqrt{1-\rho_j^2})^{-1}\exp\{-r_j^*/(2\sqrt{1-\rho_j^2})\}\int_0^{2\pi}\exp\{B_j^2/(2A_j\sqrt{1-\rho_j^2})\}R_j(\theta)d\theta$$

where

$$R_j(\theta) = \int_0^{\infty}\rho^2\exp\{-A_j(\rho+B_j/A_j)^2/(2\sqrt{1-\rho_j^2})\}d\rho.$$

Use the substitution (30) to obtain

$$R_j(\theta) = (2\sqrt{1-\rho_j^2}/A_j)^{1/2}\int_{B_j(2A_j\sqrt{1-\rho_j^2})^{-1/2}}^{\infty}\left(\frac{2\sqrt{1-\rho_j^2}}{A_j}t^2-2\frac{B_j}{A_j}\left(\frac{2\sqrt{1-\rho_j^2}}{A_j}\right)^{1/2}t+\frac{B_j^2}{A_j^2}\right)e^{-t^2}dt.$$

Integration by parts in the first term then gives

$$R_j(\theta) = (2\sqrt{1-\rho_j^2}/A_j)^{1/2}\left[(\sqrt{\pi}/2)\left(\frac{\sqrt{1-\rho_j^2}}{A_j}+\frac{B_j^2}{A_j^2}\right)\operatorname{erfc}\{B_j/(2A_j\sqrt{1-\rho_j^2})^{1/2}\}\right.$$

$$\left.-\frac{B_j}{2A_j}\left(\frac{2\sqrt{1-\rho_j^2}}{A_j}\right)^{1/2}\exp\{-B_j^2/(2A_j\sqrt{1-\rho_j^2})\}\right].$$

4. The Bivariate Exponential Density Function

In this case all double integrals appearing in (1) and (2) can be reduced to single integrals. Using the notation in (14) we have

$$\bar{N}_j(\bar{x}) = \lambda_{x_j}\lambda_{y_j}\int_0^{\infty}\int_0^{\infty}\xi[(x-\xi)^2+(y-\eta)^2]^{-1/2}\exp\{-(\lambda_{x_j}\xi+\lambda_{y_j}\eta)\}d\xi d\eta$$

$$D_j(\bar{x}) = \lambda_{x_j}\lambda_{y_j}\int_0^{\infty}\int_0^{\infty}[(x-\xi)^2+(y-\eta)^2]^{-1/2}\exp\{-(\lambda_{x_j}\xi+\lambda_{y_j}\eta)\}d\xi d\eta.$$

We consider only the first component of $\bar{N}_j(\bar{x})$ which we denote by $N_j(\bar{x})$. Let $\xi - x = \rho\cos\Theta$, $\eta - y = \rho\sin\Theta$, $\tilde{r} = x^2+y^2$, $\cos\tilde{\Theta} = -x/\tilde{r}$, $\sin\tilde{\Theta} = -y/\tilde{r}$. Thus, $\tan\tilde{\Theta} = y/x$ and $\pi \leq \tilde{\Theta} \leq 3\pi/2$.

Then we obtain

$$N_j(\bar{x}) = \lambda_{x_j}\lambda_{y_j}\left\{\left(\int_0^{\pi/2}\int_0^{\infty} + \int_{\pi/2}^{\tilde{\Theta}}\int_0^{-x/\cos\Theta} + \int_{\tilde{\Theta}}^{2\pi}\int_0^{-y/\sin\Theta}\right)\left((x+\rho\cos\Theta)\right.\right.$$
$$\left.\left.\exp\{-\lambda_{x_j}(x+\rho\cos\Theta) - \lambda_{y_j}(y+\rho\sin\Theta)\}\right)d\rho d\Theta\right\}$$
$$= \lambda_{x_j}\lambda_{y_j}\exp\{-x\lambda_{x_j} - y\lambda_{y_j}\}\left[\left(\int_0^{\pi/2}\int_0^{\infty} + \int_{\pi/2}^{\tilde{\Theta}}\int_0^{-x/\cos\Theta} + \int_{\tilde{\Theta}}^{2\pi}\int_0^{-y/\sin\Theta}\right)\left((x+\rho\cos\Theta)\right.\right.$$
$$\left.\left.\exp\{-\rho(\lambda_{x_j}\cos\Theta + \lambda_{y_j}\sin\Theta)\}\right)D\rho d\Theta\right]$$
$$= \lambda_{x_j}\lambda_{y_j}\exp\{-x\lambda_{x_j} - y\lambda_{y_j}\}\left[N_{1j}(\bar{x}) + N_{2j}(\bar{x}) + N_{3j}(\bar{x})\right]. \tag{32}$$

Now let

$$r_j^2 = \lambda_{x_j}^2 + \lambda_{y_j}^2 \qquad \Psi = \Theta - \Phi_j$$

$$\cos\Phi_j = \lambda_{x_j}/r_j \qquad \sin\Phi_j = \lambda_{y_j}/r_j.$$

Then

$$N_{1j}(\bar{x}) = \int_0^{\pi/2}\int_0^{\infty}(x+\rho\cos\Theta)\exp\{-\rho r_j\cos(\Theta-\Phi_j)\}\,d\rho d\Theta$$
$$= x\int_0^{\pi/2}\frac{d\Theta}{r_j\cos(\Theta-\Phi_j)} + \int_0^{\pi/2}\frac{\cos\Theta d\Theta}{r_j^2\cos^2(\Theta-\Phi_j)}$$
$$= \frac{x}{r_j}\int_{-\Phi_j}^{\pi/2-\Phi_j}\frac{d\Psi}{\cos\Psi} + \frac{1}{r_j^2}\int_{-\Phi_j}^{\pi/2-\Phi_j}\frac{\cos(\Psi+\Phi_j)\,d\Psi}{\cos^2\Psi}$$
$$= \frac{x}{r_j}\int_{-\Phi_j}^{\pi/2-\Phi_j}\sec\Psi\,d\Psi + \frac{1}{r_j^2}\left[\cos\Phi_j\int_{-\Phi_j}^{\pi/2\Phi_j}\sec\Psi\,d\Psi - \sin\Phi_j\int_{-\Phi_j}^{\pi/2\Phi_j}\sec\Psi\tan\Psi\,d\Psi\right]$$
$$= \left(\frac{x}{r_j} + \frac{\lambda_{xj}}{r_j^3}\right)\log\left[\frac{\lambda_{xj}}{\lambda_{yj}}\left(\frac{r_j+\lambda_{xj}}{r_j-\lambda_{yj}}\right)\right] + \frac{1}{\lambda_{xj}r_j^2}(\lambda_{yj}-\lambda_{xj})$$

and

$$N_{2j}(\bar{x}) = \int_{\pi/2}^{\tilde{\Theta}}\int_0^{x/\cos\Theta}(x+\rho\cos\Theta)\exp\{-\rho(\lambda_{xj}\cos\Theta + \lambda_{yj}\sin\Theta)\}\,d\rho d\Theta$$
$$= x\int_{\pi/2}^{\tilde{\Theta}}\frac{1-\exp\left\{\frac{x}{\cos\Theta}(r_j\cos(\Theta-\Phi_j)\right\}}{r_j\cos(\Theta-\Phi_j)}\,d\Theta$$
$$+\int_{\pi/2}^{\tilde{\Theta}}\frac{\cos\Theta\left(1-\left[\exp\left\{\frac{x}{\cos\Theta}r_j\cos(\Theta-\Phi_j)\right\}\right]\left[-\frac{x}{\cos\Theta}r_j\cos(\Theta-\Phi_j)+1\right]\right)d\Theta}{r_j^2\cos^2(\Theta-\Phi_j)}$$

$$N_{3j}(\bar{x}) = \int_{\bar{\Theta}}^{2\pi} \int_0^{-y/\sin\Theta} (x + \rho\cos\Theta)\exp\{-\rho(\lambda_{x_j}\cos\Theta + \lambda_{y_j}\sin\Theta)\}d\rho d\Theta$$

$$= x\int_{\bar{\Theta}}^{2\pi} \frac{1-\exp\left\{\dfrac{y}{\sin\Theta} r_j\cos(\Theta-\Phi_j)\right\}}{r_j\cos(\Theta-\Phi_j)} d\Theta$$

$$+ \int_{\bar{\Theta}}^{2\pi} \frac{\cos\Theta\left(1-\left[\exp\left\{\dfrac{y}{\sin\Theta} r_j\cos(\Theta-\Phi_j)\right\}\right]\left[-\dfrac{y}{\sin\Theta} r_j\cos(\Theta-\Phi_j)+1\right]\right)d\Theta}{r_j^2\cos^2(\Theta-\Phi_j)}.$$

Entirely analogous calculations give

$$D_j(x) = \lambda_{x_j}\lambda_{y_j}\left(\int_0^{\pi/2}\int_0^{\infty} + \int_{\pi/2}^{\bar{\Theta}}\int_0^{-x/\cos\Theta} + \int_{\bar{\Theta}}^{2\pi}\int_0^{-y/\sin\Theta}\right.$$

$$\left.(\exp\{-\lambda_{x_j}(x+\rho\cos\Theta) - \lambda_{y_j}(y+\rho\sin\Theta)\})\, d\rho d\Theta\right)$$

$$= \lambda_{x_j}\lambda_{y_j}\exp\{-x\lambda_{x_j} - y\lambda_{y_j}\}\left(\frac{1}{r_j}\log\left[\frac{\lambda_{x_j}}{\lambda_{y_j}}\left(\frac{r_j+\lambda_{x_j}}{r_j-\lambda_{y_j}}\right)\right]\right.$$

$$+ \int_{\pi/2}^{\bar{\Theta}} \frac{1-\exp\left\{\dfrac{x}{\cos\Theta} r_j\cos(\Theta-\Phi_j)\right\}}{r_j\cos(\Theta-\Phi_j)} d\Theta$$

$$\left. + \int_{\bar{\Theta}}^{2\pi} \frac{1-\exp\left\{\dfrac{y}{\sin\Theta} r_j\cos(\Theta-\Phi_j)\right\}}{r_j\cos(\Theta-\Phi_j)} d\Theta\right). \tag{33}$$

In order to evaluate $\psi(\bar{x})$ use the notation in (31), where now

$$z_j(\bar{x}) = \beta_j\lambda_{x_j}\lambda_{y_j}\int_0^{\infty}\int_0^{\infty}[(x-\xi)^2+(y-\eta)^2]^{1/2}\exp\{-(\lambda_{x_j}\xi + \lambda_{y_j}\eta)\}d\xi d\eta$$

$$= \beta_j\lambda_{x_j}\lambda_{y_j}\exp\{-x\lambda_{x_j} - y\lambda_{y_j}\}\left[\left(\int_0^{\frac{\pi}{2}}\int_0^{\infty} + \int_{\frac{\pi}{2}}^{\bar{\Theta}}\int_0^{\frac{-x}{\cos\Theta}} + \int_{\bar{\theta}}^{2\pi}\int_0^{\frac{-y}{\sin\Theta}}\right)\right.$$

$$\left.(\rho^2\exp\{-\rho(\lambda_{x_j}\cos\Theta + \lambda_{y_j}\sin\Theta)\})d\rho d\Theta\right]$$

$$= \beta_j\lambda_{x_j}\lambda_{y_j}\exp\{-x\lambda_{x_j} - y\lambda_{y_j}\}[z_{1j}(\bar{x}) + z_{2j}(\bar{x}) + z_{3j}(\bar{x})]. \tag{34}$$

Now we have

$$z_{1j}(\bar{x}) = \int_0^{\frac{\pi}{2}} \frac{2d\Theta}{(\lambda_{x_j}\cos\Theta + \lambda_{y_j}\sin\Theta)^3} = 2\int_0^{\frac{\pi}{2}} \frac{d\Theta}{r_j^3\cos^3(\Theta-\Phi_j)} = \frac{2}{r_j^3}\int_{-\Phi_j}^{\frac{\pi}{2}-\Phi_j} \sec^3\Psi d\Psi$$

$$= \frac{1}{r_j^2}\left(\frac{\lambda_{x_j}}{\lambda_{y_j}^2} + \frac{\lambda_{y_j}}{\lambda_{x_j}^2}\right) + \frac{1}{r_j^3}\log\left[\frac{\lambda_{x_j}}{\lambda_{y_j}}\left(\frac{r_j+\lambda_{x_j}}{r_j-\lambda_{y_j}}\right)\right]$$

$$z_{2j}(\bar{x}) = \int_{\frac{\pi}{2}}^{\tilde{\Theta}} \frac{2 - \left[\exp\left\{\frac{x}{\cos\Theta} r_j \cos(\Theta - \Phi_j)\right\}\right]\left[\left(\frac{-x}{\cos\Theta} r_j \cos(\Theta - \Phi_j) + 1\right)^2 + 1\right] d\Theta}{r_j^3 \cos^3(\theta - \Phi_j)}$$

$$z_{3j}(\bar{x}) = \int_{\tilde{\Theta}}^{2\pi} \frac{2 - \left[\exp\left\{\frac{y}{\sin\Theta} r_j \cos(\Theta - \Phi_j)\right\}\right]\left[\left(\frac{-y}{\cos\Theta} r_j \cos(\Theta - \Phi_j) + 1\right)^2 + 1\right] d\Theta}{r_j^3 \cos^3(\Theta - \Phi_j)} .$$

5. The Bivariate Symmetric Exponential Density Function

In this case all the double integrals appearing in (1) and (2) can be reduced to single integrals. Using the notation in (14) we now have

$$\bar{N}_j(\bar{x}) = (1/4)\lambda_{x_j}\lambda_{y_j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\xi}[(x-\xi)^2 + (y-\eta)^2]^{-1/2} \exp\{-\lambda_{x_j}|\xi - \theta_{x_j}| - \lambda_{y_j}|\eta - \theta_{y_j}|\} d\xi d\eta$$

$$D_j(\bar{x}) = (1/4)\lambda_{x_j}\lambda_{y_j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x-\xi)^2 + (y-\eta)^2]^{-1/2} \exp\left\{-\lambda_{x_j}|\xi - \theta_{x_j}| - \lambda_{x_j}|\eta - \theta_{y_j}|\right\} d\xi d\eta.$$

Consider only the first component of $\bar{N}_j(\bar{x})$ which we again denote by $N_j(\bar{x})$. We have

$$\begin{aligned} N_j(\bar{x}) = (1/4)\lambda_{x_j}\lambda_{y_j} \Bigg(& \int_{-\infty}^{\theta_{x_j}}\int_{-\infty}^{\theta_{y_j}} \xi |\bar{x} - \bar{\xi}|^{-1} \exp\{-\lambda_{x_j}(\theta_{x_j} - \xi) - \lambda_{y_j}(\theta_{y_j} - \eta)\} d\xi d\eta \\ & + \int_{\theta_{x_j}}^{\infty}\int_{-\infty}^{\theta_{y_j}} \xi |\bar{x} - \bar{\xi}|^{-1} \exp\{-\lambda_{x_j}(\theta_{x_j} - \xi) - \lambda_{y_j}(\eta - \theta_{y_j})\} d\xi d\eta \\ & + \int_{-\infty}^{\theta_{x_j}}\int_{\theta_{x_j}}^{\infty} \xi |\bar{x} - \bar{\xi}|^{-1} \exp\{-\lambda_{x_j}(\xi - \theta_{x_j}) - \lambda_{y_j}(\theta_{y_j} - \eta)\} d\xi d\eta \\ & + \int_{\theta_{x_j}}^{\infty}\int_{\theta_{y_j}}^{\infty} \xi |\bar{x} - \bar{\xi}|^{-1} \exp\{-\lambda_{x_j}(\xi - \theta_{x_j}) - \lambda_{y_j}(\eta - \theta_{y_j})\} d\xi d\eta \Bigg) \\ = (1/4)\lambda_{x_j}\lambda_{y_j} & (N_{1j}(\bar{x}) + N_{2j}(\bar{x}) + N_{3j}(\bar{x}) + N_{4j}(\bar{x})). \end{aligned}$$

Now let $\xi - x = \rho \cos\theta$, $\eta - y = \rho \sin\theta$, $R_j^2 = (\theta_{x_j} - x)^2 + (\theta_{y_j} - y)^2$, $\cos\theta_j = (\theta_{x_j} - x)/R_j$, $\sin\theta_j = (\theta_{y_j} - y)/R_j$. The limits of integration are determined differently depending upon whether $(\cos\theta_j, \sin\theta_j)$ is in the first, second, third, or fourth quadrant. We illustrate the procedure by calculating $N_{1j}(\bar{x})$ assuming that $(\cos\theta_j, \sin\theta_j)$ is in the first quadrant. We then have

$$\begin{aligned} N_{1j}(\bar{x}) = \exp\{-\lambda_{x_j}(\theta_{x_j} - x) - \lambda_{y_j}(\theta_{y_j} - y)\} \cdot \Bigg\{ & \int_{-\frac{\pi}{2}}^{\theta_j}\int_0^{R_j \frac{\cos\theta_j}{\cos\theta}} + \int_{\theta_j}^{\pi}\int_0^{R_j \frac{\sin\theta_j}{\sin\theta}} + \int_{-\pi}^{-\frac{\pi}{2}}\int_0^{\infty} \\ & ((x + \rho\cos\theta) \exp\{\rho(\lambda_{x_j}\cos\theta + \lambda_{y_j}\sin\theta)\}) d\rho d\theta \Bigg\} \end{aligned}$$

Now let $r_j^2 = \lambda_{x_j}^2 + \lambda_{y_j}^2$, $\cos\Phi_j = \lambda_{x_j}/r_j$, $\sin\Phi_j = \lambda_{y_j}/r_j$ to obtain

$$\begin{aligned} N_{1j}(\bar{x}) = \exp\{-\lambda_{x_j}(\theta_{x_j} - x) - \lambda_{y_j}(\theta_{y_j} - y)\} \\ \cdot \Bigg\{ & \int_{-\pi/2}^{\theta_j}\int_0^{R_j\cos\theta_j/\cos\theta} + \int_{\theta_j}^{\pi}\int_0^{R_j\sin\theta_j/\sin\theta} + \int_{-\pi}^{-\pi/2}\int_0^{\infty} \\ & ((x + \rho\cos\theta) \exp\{\rho r_j \cos(\theta - \Phi_j)\})\, d\rho d\theta. \end{aligned}$$

The integration with respect to ρ can be performed explicitly in each term to give

$$N_{1j}(\bar{x}) = \exp\{-\lambda_{xj}(\theta_{xj}-x)-\lambda_{yj}(\theta_y-y)\}$$

$$\cdot\left\{\int_{-\pi/2}^{\theta_j}\left[x\left(\frac{\left[\exp\left\{R_j\frac{\cos\theta_j}{\cos\theta}r_j\cos(\theta-\Phi_j)\right\}\right]-1}{r_j\cos(\theta-\Phi_j)}\right)\right.\right.$$

$$\left.+\frac{\cos\theta}{r_j^2\cos^2(\theta-\Phi_j)}\left(1-\left(-R_j\frac{\cos\theta_j}{\cos\theta}r_j\cos(\theta-\Phi_j)+1\right)\exp\left\{\frac{R_j\cos\theta_j}{\cos\theta}r_j\cos(\theta-\Phi_j)\right\}\right)\right]d\theta$$

$$+\int_{\theta_j}^{\pi}\left[x\left(\frac{\left[\exp\left\{R_j\frac{\sin\theta_j}{\sin\theta}r_j\cos(\theta-\Phi_j)\right\}\right]-1}{r_j\cos(\theta-\Phi_j)}\right)\right.$$

$$\left.\left.+\frac{\cos\theta}{r_j^2\cos^2(\theta-\Phi_j)}\left(1-\left(-R_j\frac{\sin\theta_j}{\sin\theta}r_j\cos(\theta-\Phi_j)+1\right)\exp\left\{R_j\frac{\sin\theta_j}{\sin\theta}r_j\cos(\theta-\Phi_j)\right\}\right)\right]d\theta\right\}$$

$$+\frac{xr_j^2-\lambda_{xj}}{{}_jr^3}\log\left(\frac{\lambda_{xj}}{\lambda_{yj}}\left[\frac{r_j+\lambda_{xj}}{r_j-\lambda_{yj}}\right]\right)-\frac{\lambda_{yj}-\lambda_{xj}}{\lambda_{xj}r_j^2}. \qquad (35)$$

The last two terms have been evaluated by transforming the range of integration to $\left[0, \frac{\pi}{2}\right]$ and then proceeding as in (33) and (34). Similar expressions are obtained for $N_{2j}(\bar{x})$, $N_{3j}(\bar{x})$, $N_{4j}(\bar{x})$ and for the analogous terms in $D_j(\bar{x})$. The objective function $\psi(\bar{x})$ in (1) is evaluated analogously except that the integrand with respect to ρ is of the form $\rho^2\exp(A\rho)$, with A independent of ρ.

6. Numerical Results and Discussion

For $\phi_j(\bar{x}_j)$ given by (A), (B), and (C) we have performed a large number of numerical computations to find the minimizing P, and $\bar{\psi}$. In [6], we presented an example illustrating the locally linear convergence to P for the case (A), bivariate normal density functions. We do the same here for the case (B), exponential density functions, and the case (C), symmetric exponential density functions. We also show how the use of Steffensen's iteration [4], accelerates the convergence. In order to emphasize the contrast between this probabilistic problem and the corresponding deterministic one, we indicate those cases in which $\bar{x} \notin \text{co}\{\bar{\mu}_k\}_{k=1}^N$ with an asterisk. Comparisons are also made between $\bar{\psi}$, the bound for $\bar{\psi}$ from (4), and ψ BOUND from (5).

Problem No. 1 below exhibits a typical set of data used in one of a series of problems studied numerically for case (B), the bivariate exponential density function. In this problem $N=20$, and the $\phi_j(\bar{x}_j)$ for $j=1, 2, \ldots, N$ were taken as bivariate exponential density functions whose values of λ_{xj} and λ_{yj} were generated randomly. The weights β_j were also generated randomly. The iteration was terminated when $|x_{n+1}-x_n| \leq 10^{-4}$ and $|y_{n+1}-y_n| \leq 10^{-4}$. $\bar{x}_0$ was chosen to be

$$\frac{\sum_{j=1}^{N}\beta_j\bar{\lambda}_j^{-1}}{\sum_{j=1}^{N}\beta_j}$$

where $\bar{\lambda}_j^{-1}=(\lambda_{xj}^{-1}, \lambda_{yj}^{-1})$. M in table 3 is the number of iterations required to achieve the desired accuracy. The computer used for all calculations was the CDC Cyber 72.

Problem No. 1

λ_{xj}:	1.36	3.56	13.59	10.33	6.30	5.37	15.76	16.43	4.78	8.94
	6.68	17.18	7.39	15.90	0.31	7.03	19.88	12.86	14.85	10.94
λ_{yj}:	7.97	16.21	8.78	8.82	15.01	2.50	0.18	10.35	18.61	8.07
	18.00	7.36	7.87	2.90	9.40	11.18	10.76	6.54	13.61	12.66
β_j:	9.06	8.76	9.85	7.94	2.82	8.83	2.39	2.47	9.41	5.70
	7.36	9.00	0.94	0.50	1.27	0.13	6.49	5.72	4.28	9.22

The method of numerical solution employed is to use (2). This is discussed in more detail in [6]. Here we simply wish to illustrate the same kind of linear local convergence for the bivariate exponential density function as was shown in [6] for the bivariate normal density function. In table 2, we show this local convergence.

TABLE 2. *Local convergence—bivariate exponential density function*

n	x_n	y_n	$x_n-\alpha$	$y_n-\beta$	$\lvert\bar{x}_n-\bar{\alpha}\rvert$	$\frac{\lvert\bar{x}_{n+1}-\bar{\alpha}\rvert}{\lvert\bar{x}_n-\bar{\alpha}\rvert}$
1	0.326081	0.144027	−0.071494	0.049108	0.08673	0.5442
2	.354470	.114142	−.043105	.019223	.04720	.5983
3	.370288	.102158	−.027292	.007239	.02824	.6317
4	.379962	.097750	−.017613	.002831	.01784	.6373
5	.386272	.096122	−.011303	.001203	.01136	.6356
6	.390370	.095491	−.007205	.000572	.00722	.6191
7	.393015	.095225	−.004560	.000306	.00447	.6398
8	.394707	.095100	−.002868	.000181	.00286	.6049
9	.395790	.095028	−.001785	.000109	.00173	.6300
10	.396482	.094985	−.001093	.000066	.00109	.5963
11	.396925	.094958	−.000650	.000039	.00065	.5692
12	.397207	.094941	−.000368	.000022	.00037	.5135
13	.397387	.094930	−.000188	.000011	.00019	.3684
14	.397502	.094923	−.000073	.000004	.00007	

$\bar{\alpha}=(0.397575, 0.094919)$.

In table 3 we indicate for 40 randomly generated problems the values of $\bar{\psi}$, the bound on $\bar{\psi}$ from (4), ψ BOUND from (5), and whether or not $\bar{x}$ is in the convex hull of the means. An asterisk next to the value of $\bar{\psi}$ indicates that $\bar{x}$ is not in the convex hull of the means of distributions. The closeness of the bound is probably strongly dependent upon the nature of the data. For these randomly generated problems, the bounds are not very close.

TABLE 3. *Bounds on objective function—exponential density*

N	M	$\bar{\psi}$	Bound for $\bar{\psi}$ from (4)	$\bar{\psi}$ BOUND from (5)
30	16	29.114	173.880	242.628
30	12	12.079	100.406	140.618
30	14	5.284*	55.537	75.538
30	14	25.792	202.893	284.551
30	6	12.656	77.019	106.534
30	13	21.548	347.880	490.120
30	12	17.742	118.154	165.042
30	10	9.856	81.457	112.372
30	10	7.350	61.793	83.795

TABLE 3. *Bounds on objective function—exponential density*—Continued

N	M	ψ	Bound for ψ from (4)	ψ BOUND from (5)
30	10	8.216	79.363	110.170
20	14	5.927	53.949	74.012
20	14	26.725	151.732	212.956
20	14	8.522	68.464	95.960
20	11	3.125	39.150	53.083
20	20	23.835	168.143	236.237
20	10	4.149	51.055	70.417
20	8	9.196*	54.956	75.920
20	14	22.643	344.637	486.403
20	9	2.400	25.562	34.767
20	6	16.436	105.702	147.579
10	10	1.121*	12.074	15.948
10	11	3.318*	21.399	29.313
10	9	0.989*	12.832	16.929
10	10	5.497*	47.132	65.906
10	9	1.624	20.033	27.293
10	14	1.382*	13.027	17.314
10	10	4.314	28.651	39.116
10	14	5.008	43.205	59.725
10	15	1.140*	10.676	14.489
10	13	1.773	24.635	34.190
5	15	2.271*	14.688	20.107
5	9	1.670*	26.527	36.952
5	7	0.240*	3.334	4.015
5	13	21.036	106.369	150.094
5	18	2.140	13.529	18.449
5	8	1.075*	18.091	25.193
5	14	2.423*	13.535	18.639
5	28	4.310	26.844	37.449
5	17	1.158*	12.988	16.715
5	15	1.438*	8.857	12.148

Problem No. 2 below exhibits a typical set of data for the problems of case (C), the bivariate symmetric exponential function. In this problem $N=10$, and the $\phi_j(\bar{x}_j)$ for $j=1, 2, \ldots, N$ were taken as bivariate symmetric exponential density functions whose values of λ_{x_j}, λ_{y_j}, θ_{x_j}, θ_{y_j} were generated randomly. The weights β_j were similarly generated. The iteration was terminated when $|x_{n+1}-x_n| \leq 10^{-4}$ and $|y_{n+1}-y_n| \leq 10^{-4}$. $\bar{x}_0$ was chosen to be:

$$\frac{\sum_{j=1}^{N} \beta_j \bar{\theta}_j}{\sum_{j=1}^{N} \beta_j}$$

where $\bar{\theta}_j=(\theta_{x_j}, \theta_{y_j})$. M in table 5 is the number of iterations required to achieve the desired accuracy.

Problem No. 2

λ_{x_j}:	1.36	17.53	8.82	5.37	4.78	18.61	6.68	18.00	2.90	7.03
λ_{y_j}:	7.97	13.59	15.89	2.50	16.43	18.82	18.00	7.39	1.01	11.18
θ_{x_j}:	9.06	4.39	3.15	8.83	5.18	4.47	7.36	3.94	0.15	0.13
θ_{y_j}:	1.78	9.85	7.51	7.88	2.47	4.04	8.59	0.94	4.70	9.94
β_j:	8.11	5.17	2.82	0.09	2.39	5.70	3.68	7.95	1.27	5.38

Local convergence for Problem No. 2 is exhibited in table 4. In table 5 we indicate the bounds and whether or not the solution is in the convex hull of the means in the same fashion as we did for the bivariate symmetric exponential density function. It is similar to table 3. For the problems generated for the symmetric exponential case we never found one in the convex hull of the means.

TABLE 4. *Local convergence—bivariate symmetric exponential density function*

n	x_n	y_n	$x_n-\alpha$	$y_n-\beta$	$\lvert\bar{x}_n-\bar{\alpha}\rvert$	$\frac{\lvert\bar{x}_{n+1}-\bar{\alpha}\rvert}{\lvert\bar{x}_n-\bar{\alpha}\rvert}$
1	0.01668185	0.04066205	0.00462450	0.03773900	0.0380213	0.3519
2	.01158224	.01629392	−0.00047511	.01337087	.0133793	.3051
3	.01210797	.00700439	.00005062	.00408134	.00408165	.3864
4	.01255962	.00441807	.00050227	.00149502	.00157714	.5018
5	.01253156	.00355670	.00047421	.00063365	.000791446	.5563
6	.01238560	.00321650	.00032825	.00029345	.000440296	.5251
7	.01224484	.00305836	.00018749	.00013531	.000231217	.4034
8	.01213578	.00297353	.00007843	.00005048	.0000932711	

$\bar{\alpha}=(0.01205735, 0.00292305)$.

TABLE 5. *Bounds on objective function—symmetric exponential density*

N	M	$\bar{\psi}$	Bound for $\bar{\psi}$ from (4)	$\bar{\psi}$ Bound from (5)
30	13	38.195*	485.942	533.053
30	19	36.889*	609.044	699.347
30	11	31.590*	623.526	696.993
30	14	15.391*	513.657	569.532
30	13	26.269*	850.293	978.537
30	11	3.362*	527.453	596.923
30	13	15.910*	754.403	823.259
30	28	54.418*	553.564	611.917
30	10	56.257*	723.158	801.155
30	12	7.493*	482.261	544.145
20	16	0.02035*	324.837	349.834
20	17	119.638*	503.372	565.895
20	10	5.548*	312.673	350.797
20	12	10.965*	369.864	420.667
20	16	17.922*	469.078	527.044
20	25	199.676*	682.171	762.248
20	13	72.808*	553.628	631.301
20	21	2.169*	215.444	241.296
20	9	4.340*	452.851	483.389
20	13	0.45620*	342.029	376.407
10	8	2.407*	178.300	193.276
10	17	29.185*	308.299	366.341
10	15	4.695*	107.535	117.199
10	9	0.47852*	177.145	187.936
10	12	5.742*	237.441	254.580
10	19	25.345*	205.296	240.394
10	12	9.546*	186.705	203.622
10	12	1.429*	140.736	150.786
10	20	2.901*	155.144	178.605
10	5	0.77443*	286.310	307.411
5	12	9.236*	93.932	124.836
5	25	0.00133*	49.655	55.333
5	4	1.951*	85.311	94.246
5	19	14.127*	216.730	273.959
5	8	0.00000*	88.316	93.744
5	8	2.465*	60.637	67.276
5	17	3.493*	100.387	110.473
5	13	1.301*	80.547	85.485
5	23	30.839*	69.109	79.990
5	13	0.00000*	31.311	35.560

6.1. Steffensen's Iteration

A well known method for accelerating the convergence of linearly convergent iterative schemes is Steffensen's iteration [4]. In the case when the unknown vector is two dimensional, Steffensen's iteration, applied to the scheme in (2), is described as follows (see [4], p. 115 for a derivation):

Choose a vector $\bar{x}^{(0)}=(x^{(0)}, y^{(0)})$, and construct the sequence of vectors $\{\bar{x}^{(k)}\}$ by the rule: for each $k=0, 1, 2, \ldots$, set $\bar{x}_0=\bar{x}^{(k)}$, and calculate $\bar{x}_1, \bar{x}_2, \bar{x}_3$ from

$$\bar{x}_{n+1}=\bar{H}(x_n) \qquad n=0, 1, 2.$$

Let the matrix X_n be defined by

$$X_n=\begin{pmatrix} x_n & x_{n+1} \\ y_n & y_{n+1} \end{pmatrix}$$

and let Δ denote the forward difference operator. The next iterate is given by

$$\bar{x}^{(k+1)}=\bar{x}_0-\Delta X_0(\Delta^2 X_0)^{-1}\Delta\bar{x}_0.$$

In order to test the utility of Steffensen's iteration in reducing the amount of calculation to converge to any given degree of tolerance, ten, 20 destination problems and ten, 10 destination problems were run on the CDC Cyber 72 both with and without the use of Steffensen's iteration

TABLE 6. *Effect of Steffensen's iteration on convergence*

Number of destinations	Number of iterations	Number of iterations (Steffensen)
20	14	7
20	14	13
20	16	6
20	16	13
20	17	15
20	14	6
20	12	5
20	15	5
20	15	9
20	13	6
20	14	6
10	15	12
10	15	7
10	16	6
10	16	8
10	18	9
10	12	10
10	17	10
10	13	8
10	25	13
10	16	7

technique. The number of iterations given in table 6 is the *total* number of iterations, i.e., the number reported using Steffensen's iteration includes the three iterations required to use the method. The problems all used a bivariate normal density function. The average ratio of the number of iterations with the Steffensen method to the number of iterations without the method for the 20 problems of table 6 was 0.571. Hence, it can be readily seen that it is quite advantageous to use the Steffensen iteration method when performing these calculations.

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Real, 3 × 3, D-Stable Matrices*

Bryan E. Cain**

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We characterize the 3×3 real D-stable Matrices.

Key words: Arc-connectedness; Carlson-Johnson conjecture; D-stable; positive stable matrix.

We give a complete description of the 3×3, real, D-stable matrices and thereby extend one of the main theorems in [2].[1] One nice feature of our characterizing conditions is that the requisite invariance under action by the multiplicative group of diagonal matrices with positive diagonal entries and the requisite invariance under inversion are clear. Our result is more evidence that the set of D-stable matrices is complicated, and we hope that it will assist the search for a general description, by helping with the formulation of conjectures and by providing counterexamples. For more information on this extensively studied problem see [3].

We call the square matrix M (positive) *stable* provided that $\mathrm{Re}(\lambda)>0$ for every eigenvalue λ of M. And M is *D-stable* provided that MD is stable for every diagonal matrix D whose diagonal entries are positive.

Let $M=(m_{ij})$ be a 3×3 matrix with real entries. The principal minors of M will be denoted: $a=m_{11}$, $b=m_{22}$, $c=m_{33}$, $A=m_{22}m_{33}-m_{32}m_{23}$, $B=m_{11}m_{33}-m_{31}m_{13}$, $C=m_{11}m_{22}-m_{21}m_{12}$, $\delta=\det(M)$. We shall say that a and A, b and B, c and C, and 1 and δ are supplementary principal minors of each other. We say that M is of type 1 if some principal minor of M vanishes without its supplement vanishing also. Otherwise M is of type 2.

Let $\Delta=\sqrt{aA}+\sqrt{bB}+\sqrt{cC}$.

THEOREM: M *is* D*-stable if and only if*

(i) a, b, c, A, B, C *are non-negative,*
(ii) $a+b+c$, $A+B+C$, δ *are positive, and*
(iii) $\delta\leq\Delta^2$ *if* M *is of type 1;*
$\delta<\Delta^2$ *if* M *is of type 2.*

PROOF: Let $D=\mathrm{diag}(x, y, z)$. The Routh-Hurwitz Theorem [5], applied to the characteristic polynomial of MD (cf. [2]), says that MD is stable if and only if

$$(\alpha)\quad ax+by+cz>0$$
$$(\beta)\quad (ax+by+cz)(Ayz+Bxz+Cxy)-xyz\delta>0$$
$$(\gamma)\quad xyz\delta>0.$$

Thus M is D-stable if and only if (α), (β), (γ) hold for all $x, y, z>0$. That (α) holds for all $x, y, z>0$ is clearly equivalent to "$a, b, c\geq0$ and $a+b+c>0$". That (γ) holds is equivalent to "$\delta>0$". When (α) and (γ) hold then (β) implies that $Ayz+Bxz+Cxy>0$. If M is D-stable this is true for all

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* An invited paper.
**Present address: technische Universität München 8 München 2, Arcisstraße 21.
[1] Figures in brackets indicate the literature references at the end of this paper.

$x, y, z > 0$, and so $A, B, C \geq 0$ and $A+B+C > 0$. To summarize: M is D-stable implies (i) and (ii), and (i), (ii) imply that (α), (γ) hold for all $x, y, z > 0$ (cf. [2]). We pause here to prove:

LEMMA: *Suppose that the principal minors of* M *satisfy (i) and (ii).*

Let $f(x, y, z) = (ax+by+cz)\,(Ayz+Bxz+Cxy)/xyz$ *and let* J *denote the infimum of* $f(x, y, z)$ *for* $x, y, z > 0$. *Then*

(1) $J = \Delta^2$

(2) $f(x, y, z) > \Delta^2$ *for all* $x, y, z > 0$ *if* M *is of type 1*

(3) $f(x, y, z) = \Delta^2$ *for some* $x, y, z > 0$ *if* M *is of type 2.*

PROOF: A bit of algebra shows:

$$(*)\quad xyz(f(x, y, z) - \Delta^2) = z(\sqrt{aB}\,x - \sqrt{bA}\,y)^2 + y(\sqrt{aC}\,x - \sqrt{cA}\,z)^2 + x(\sqrt{bC}\,y - \sqrt{cB}\,z)^2.$$

Thus $J \geq \Delta^2$, and when M is of type 2 we can verify (1) and (3) by selecting $x, y, z > 0$ so that the quadratic terms vanish. Indeed, if M is of type 2 then the coefficients of each quadratic term either both vanish or are both nonzero. Thus, a quadratic term which is not identically zero will be zero for appropriate positive values of its variables. If none of the quadratic terms is identically zero then all will vanish if $x = \sqrt{cA/aC}$, $y = \sqrt{cB/bC}$, $z = 1$. In all other cases the choice of $x, y, z > 0$ is easy because at least two of the quadratic terms must vanish identically.

For M's of type 1 we have yet to prove (1) and (2). In this case at least one nonzero principal minor has a vanishing supplement and we change the notation if necessary so that $cC = 0$, $c + C > 0$. We define T so that equation (*) divided by xyz is $f(x, y, z) - \Delta^2 = [(\sqrt{aB}\,x - \sqrt{bA}\,y)^2/xy] + T$. If $c = 0$ and $C > 0$ then $T = C(ax+by)/z$, and if $c > 0$ and $C = 0$ then $T = cz(A/x + B/y)$. Thus in both cases (i) and (ii) imply that $T > 0$ for all $x, y, z > 0$ and (2) follows. Furthermore, in both cases, if we fix $x, y > 0$ we can manipulate $z > 0$ to make T small. Thus we can establish (1) by finding $x, y > 0$ for which the quadratic term divided by xy is small. If $aB = bA = 0$ this is trivial. If $aBbA \neq 0$ we set $x = \sqrt{bA}$, $y = \sqrt{aB}$ so that the quadratic term vanishes. In the only other case exactly one of the coefficients $\sqrt{aB}$, $\sqrt{bA}$ is nonzero. We then set the variable it multiplies equal to 1 and makes the other of the two variables x, y very large.

We now return to the proof of the theorem. If M is D-stable then equation (β) holds for all $x, y, z > 0$. By the lemma that implies that $\Delta^2 \geq \delta$ if M is of type 1 and $\Delta^2 > \delta$ if M is of type 2. In other words (iii) holds. Conversely, when (iii) holds the lemma shows that for all $x, y, z > 0$, $xyz\,\delta \leq zyx\,\Delta^2 < xyz\,f(x, y, z)$ if M is of type 1, $xyz\,\delta < xyz\,\Delta^2 \leq xyz\,f(x, y, z)$ if M is of type 2. Thus equation (β) holds for all $x, y, z > 0$ and M is D-stable.

COROLLARY: *The real* 3×3 *matrix* M *is* D*-stable if and only if* M + D *is* D*-stable for all diagonal* D *with nonnegative entries.*

PROOF: The converse is trivial, take $D = 0$.

Now assume that M is D-stable. Since $M + D$ may be viewed as the result of three successive pertubations of M by diagonal matrices each having at most one nonzero entry, there is no loss in assuming D has only one nonzero entry. Moreover, we may assume that $D = \mathrm{diag}(x, 0, 0)$ with $x > 0$. Since $M + D = x\left[\frac{1}{x}M + \mathrm{diag}(1, 0, 0)\right]$ and $\frac{1}{x}M$ is D-stable, it will be sufficient if we can prove that $M' = M + \mathrm{diag}(1, 0, 0)$ is D-stable if M is. Let $a', b', \ldots, C', \delta', \Delta'$ be defined with respect to M' just as $a, b, \ldots, \Delta$ were with respect to M. Then $a' = a+1$, $b' = b$, $c' = c$, $A' = A$, $B' = B + c$, $C' = C + b$, $\delta' = \delta + A$. Thus M' has properties (i) and (ii). Since $(\Delta')^2 - \delta' \geq \Delta^2 - \delta + 2bc$ and $2bc$,

$\Delta^2-\delta$ are nonnegative, we are sure that M' is D-stable except in the case (which we will show never occurs) that $\Delta^2-\delta=bc=0$ and M' is of type 2. In that case a principal minor of M' vanishes if and only if its supplement does. Then $B'=B+c>0 \Rightarrow b'=b>0 \Rightarrow C'=C+b>0 \Rightarrow c'=c>0$. So the assumption $B'>0$ contradicts $bc=0$. Similarly the assumption $C'>0$ also contradicts $bc=0$. Thus $B'=C'=0$, that is $b=c=B=C=0$. Conditions (i) and (ii) now imply that a, A are positive. This means that M is of type 2 and since M is D-stable $\Delta^2-\delta>0$, a contradiction.

The preceding corollary is more evidence for the Carlson-Johnson Conjecture [1] that if M is $n\times n$ and D-stable then $M+D$ is stable for all diagonal D with positive diagonal entries.

COROLLARY: *The set of 3×3 real* D*-stable matrices is arc-connected.*

PROOF: If M is D-stable, then $(1-t)M+t\mathrm{I}$ for $0\leq t\leq 1$ is, by the preceding corollary, an arc of D-stable matrices connecting M to I.

Both corollaries hold in the 1×1 and 2×2 cases also.

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Transient Solutions for Stratified Fluid Flows*

David W. Fox**

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The solution to an initial value problem for the flow of a buoyant inviscid incompressible fluid in three-space is given in terms of a fundamental solution. The initial values and the distribution of source strength can be quite general subject only to mild restrictions on smoothness. The fundamental solution and its associated velocity and displacement fields are given explicitly and in some detail.

Key words: Fluid dynamics; fundamental solution, initial value problem; partial differential equations; stratified flow, internal waves, buoyant flows.

Introduction

This article presents a solution to an initial value problem in the theory of stratified fluids. Thus it is a contribution to the theory of buoyant flow and internal gravity waves and also to the closely related topic of flows in rotating fluids.[1] A system of two linear second order equations in three space dimensions and time govern the problem. The equations describe the flow of an inviscid incompressible fluid in which an arbitrary distribution of source strength is prescribed. The approximations that the density and the buoyancy frequency are constant are used. A fundamental solution of the system is used to solve the initial value problem in terms of the source distribution and the initial conditions. The fundamental solution is given explicitly; and a number of its properties are noted, including its velocity and displacement fields.

Section 1 gives a summary sketch of the system of equations governing the flow. They result from a first order perturbation from static equilibrium of the conservation equations for momentum, density, and mass. Needless to say, these equations agree in all respects with those given in standard works such as [8].[2] Two technical differences are worth noting. First, the unknown functions in the system with which we work differ somewhat from those most often seen; and second, we prefer to deal with a pair of second order differential equations rather than combining them into a single fourth order equation. This section concludes with the statement of an appropriate set of initial conditions for the problem.

We start section 2 by writing down the full initial-value problem in the exact form we need. Then in section 2.1 we give the fundamental solution and an interpretation of it as an impulsive injection of fluid; the corresponding velocity field is also presented. A verification of the solution is sketched. In section 2.2 the solution of the problem with a prescribed source distribution but zero initial conditions is given. Then the initial conditions are accounted for, and everything is put together. Some of the properties of the solutions are verified in this section, but other technical details will appear elsewhere [3].

Properties of the solutions are examined in section 3. The structure of the fundamental solution shows a natural decomposition of the flow into two parts, one the flow of nonbuoyant ideal fluid of constant density and the other a modification that gives the influence of buoyancy. The first part

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** Present address: Applied Physics Laboratory, The Johns Hopkins University, Laurel, Maryland 20810.
[1] These topics are in rapid development. For recent work the reader should look at the last several years of the Journal of Fluid Mechanics, Geophysical Fluid Dynamics, Physics of Fluids, and the references given there.
[2] Figures in brackets indicate the literature references at the end of this paper.

responds immediately to the injection, the second is slow and long lasting. We give especially simple expressions for the fundamental solution directly above and below the singular point and in the horizontal plane with it.

The solutions of the initial-value problem are then examined in light of the properties of the fundamental solution. We note that, in addition to an oscillating source-like effect, the initial conditions produce a term that rings without changing spatial distribution. Under certain conditions this can be the total effect of the initial conditions. In other circumstances, when the initial conditions are confined to a bounded region, we show that the influence of the initial conditions dies off at large distances faster than that of a source distribution with nonzero total strength.

The last section discusses very quickly the displacements that result from the fundamental solution. These are purely radial and are found from an exact integration of the velocities.

A summary of portions of this work has been presented in [2].

1. The Equations of Motion

We start from a usual system of equations for an inviscid incompressible fluid in a constant gravity field. These state the conservation of momentum, density (incompressibility), and mass:

$$\rho D\mathbf{u}/Dt + \text{grad } p + g\rho\mathbf{k} = 0, \tag{1.1}$$

$$D\rho/Dt = 0, \tag{1.2}$$

$$\rho \text{ div } \mathbf{u} = \mu, \tag{1.3}$$

where $\mathbf{k}$ is the vertical unit vector (in the x_3-direction), and μ is a prescribed distribution of source strength. The rest of the notation is standard. These equations are to be linearized about a state of stable static equilibrium described by $\mathbf{u} = \mathbf{u}_0 = 0$, $\rho = \rho_0(x_3)$, and $p = p_0 = -g \int_0^{x_3} \rho_0(z)dz$. For stability ρ_0 is required to be positive and monotone decreasing with x_3. The perturbed flow is taken in the form $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, $\rho = \rho_0 + \tilde{\rho}$, and $p = p_0 + \tilde{p}$. These are put in (1.1), (1.2), and (1.3), and quantities of order higher than the first are discarded to obtain the linear equations

$$\rho_0 \partial\tilde{\mathbf{u}}/\partial t + \text{grad } \tilde{p} + g\tilde{\rho}\mathbf{k} = 0, \tag{1.4}$$

$$\partial\tilde{\rho}/\partial t + \tilde{u}_3 d\rho_0/dx_3 = 0, \tag{1.5}$$

$$\rho_0 \text{ div } \tilde{\mathbf{u}} = \mu. \tag{1.6}$$

Now we drop the $\tilde{}$'s, and henceforth all the quantities will be the perturbations except for ρ_0, which designates the prescribed equilibrium density. From the horizontal components of (1.4) it follows that the vertical component of vorticity does not change with time so that for a variety of flows, including all motions that start from rest, the horizontal velocity components derive from a potential. We assume this to be the case. For convenience we write this in the form

$$u_\alpha = \frac{1}{\rho_0} \frac{\partial\varphi}{\partial x_\alpha}, \qquad \alpha = 1, 2. \tag{1.7}$$

Also from (1.4) and (1.7) it follows that p can be expressed by

$$-p = \partial\varphi/\partial t, \tag{1.8}$$

and that the equation of motion in the vertical direction takes the form

$$\frac{\partial}{\partial t}\left(\rho_0 u_3 - \frac{\partial \varphi}{\partial x_3}\right) + g\rho = 0. \tag{1.9}$$

The equations (1.9) and (1.5) imply

$$\frac{\partial^2}{\partial t^2}\left(u_3 - \frac{1}{\rho_0}\frac{\partial \varphi}{\partial x_3}\right) - \frac{g}{\rho_0}\frac{d\rho_0}{dx_3}u_3 = 0.$$

and (1.6) takes the form

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \rho_0 \frac{\partial u_3}{\partial x_3} = \mu.$$

Now we introduce the notation n and w^*. The *buoyancy frequency* n is defined by $n^2 \equiv -(g/\rho_0)d\rho_0/dx_3$, and w^* is given by

$$w^* \equiv u_3 - \frac{1}{\rho_0}\frac{\partial \varphi}{\partial x_3}. \tag{1.10}$$

We write the preceding pair of equations in the more convenient forms

$$\left(\frac{\partial^2}{\partial t^2} + n^2\right) w^* + \frac{n^2}{\rho_0}\frac{\partial \varphi}{\partial x_3} = 0, \tag{1.11}$$

and

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \rho_0 \frac{\partial}{\partial x_3}\left(\frac{1}{\rho_0}\frac{\partial \varphi}{\partial x_3}\right) + \rho_0 \frac{\partial w^*}{\partial x_3} = \mu. \tag{1.12}$$

Note that in general n^2 and ρ_0 depend on x_3.

The coupled system (1.11) and (1.12) governs the quantities φ and w^*, from which in turn the eqs (1.7), (1.8), (1.9) and the definition (1.10) determine the velocities, density, and pressure. However the system must be completed by appropriate boundary conditions for φ and w^* on the prescribed boundaries, a surface condition on free surfaces, and by initial conditions.

Here we shall be concerned with the situation in which all of the boundaries are far away from the region of interest and their influence can be neglected. In this case only initial conditions remain to be considered. These fall on w^* alone since in (1.11) and (1.12) w^* is the only quantity to be differentiated with respect to t. They can be prescribed by

$$w^*(x, 0^+) = f(x) \qquad \text{and} \qquad \frac{\partial w^*}{\partial t}(x, 0^+) = g(x) \tag{1.13}$$

where f and g are given functions. The second of (1.13) is equivalent to a statement of the initial perturbation density as can be seen from (1.9) and (1.10).

It may seem strange at first to give initial conditions on just w^* and $\partial w^*/\partial t$, but this is precisely what the system (1.11) and (1.12) requires. This is to be compared with a similar circumstance in the linear theory of surface gravity waves on an ideal fluid in which the initial values of the potential and its time derivative on the free surface are all that is required.[3] In each case the reason is much

[3] See [7] Chapter 6 for a clear exposition.

the same: an equation of motion, in our case (1.12), determines the remainder of the solution at $t=0^+$ from what is given.

2. An Initial Value Problem

Here we present a solution to a general initial-value problem for a fluid in which the density and the buoyancy frequency are treated as constant, i.e., the flow is described by (1.11) and (1.12) in which ρ_0 and n are taken to be constants. The region of the solution is all of three-space.

We shift the time scale to $\bar{t}=nt$, set $\bar{g}=g/n$, note that $\bar{n}=n/n=1$, and then drop the $^-$'s for simplicity; the problem is thereby put in the form

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+1\right)w^*+\frac{1}{\rho_0}\frac{\partial\varphi}{\partial x_3}&=0,\\ \nabla^2\varphi+\rho_0\frac{\partial w^*}{\partial x_3}&=\mu,\end{aligned}\right\}\quad x,\ t\in R^3\times(0,\infty) \tag{2.1}$$

where ∇^2 is the three-dimensional Laplacian. The initial conditions become

$$w^*(x,0^+)=f(x),\qquad \frac{\partial w^*}{\partial t}(x,0^+)=g(x). \tag{2.2}$$

Our approach to the solution of the problem stated by (2.1) and (2.2) is the determination and systematic use of a *fundamental solution* of (2.1).

The importance of a fundamental solution derives from the fact that for the equations and region for which it is valid, it reduces the solution of inhomogeneous initial-value problems to integration. Further, by a variety of more-or-less standard devices it often can be modified to apply to a number of other regions. Equally important, an analysis of the behavior of a fundamental solution can yield considerable information on the properties of whole classes of solutions, any of which might be difficult to obtain explicitly. In addition, it is a usual starting point for a formulation of initial-boundary-value problems in terms of integral equations.

2.1. A Fundamental Solution

A fundamental solution for the problem stated by (2.1) and (2.2) is a singular solution $\hat{\varphi}$, $\hat{w}^*$ of (2.1) that gives the effect at x, t of a disturbance at y, τ. Here the disturbance can be interpreted as an impulsive injection at time τ of a unit mass of fluid of density ρ_0 at the point y. The body of fluid is supposed to be at rest prior to the injection. Since the fundamental solution will depend only on $x-y$ and $t-\tau$, we need to consider only $y=0$ and $\tau=0$ for the moment.

The fluid is at rest prior to $t=0$, and consequently both $\hat{\varphi}$ and $\hat{w}^*$ will be identically zero for $t<0$. For $t\geqslant 0$ our fundamental solution is given by

$$\begin{aligned}\hat{\varphi}(x,t)&=\frac{-1}{4\pi|x|}\left\{\delta(t)+\frac{\partial}{\partial t}J_0(t\cos\theta)+[J_0(t\cos\theta)]^*\left[\frac{J_1(t)}{t}\right]\right\}\\ \hat{w}^*(x,t)&=\frac{-1}{4\pi\rho_0|x|^2}\left\{\frac{\partial}{\partial t}tJ_1(t\cos\theta)+[tJ_1(t\cos\theta)]^*\left[\frac{J_1(t)}{t}\right]\right\}\end{aligned} \tag{2.3}$$

where $|x|=(x_1^2+x_2^2+x_3^2)^{1/2}$, $\delta(t)$ is the Dirac δ-measure concentrated at zero, and $\cos\theta=x_3/|x|$.

The symbol * indicates convolution, e.g., $l(t)*m(t) \equiv \int_0^t l(\tau)m(t-\tau)d\tau$; and J_0 and J_1 are Bessel functions of the first kind. The *fundamental velocities* $\hat{\mathbf{u}}$ are those corresponding by (1.7) and (1.10) to the fundamental solution (2.3). These formulas yield by a direct calculation.

$$\hat{\mathbf{u}} = \frac{x}{4\pi\rho_0|x|^3}\left\{ \delta(t) + \frac{\partial^2}{\partial t^2} t J_0(t\cos\theta) + \left[\frac{\partial}{\partial t} t J_0(t\cos\theta) \right] * \left[\frac{J_1(t)}{t} \right] \right\} \tag{2.4}$$

In the computation it is economical to differentiate the quantity in braces in $\hat{\varphi}$ with respect to $\cos\theta$ and then $\cos\theta$ with respect to each of the space variables.

We give elsewhere [3] a detailed and mathematically complete demonstration of the validity of the fundamental solution and the representation formulas that make use of it. Here we give only a very brief sketch.

Note first that $\hat{\varphi}$, $\hat{w}^*$, and $\hat{\mathbf{u}}$ given by (2.3) and (2.4) have the correct coordinate singularities for a fundamental solution [1]. It is quite easy to show that $\hat{\varphi}$ and $\hat{w}^*$ do satisfy (2.1) with μ equal to zero for any nonzero x. In fact, the first of (2.1) is demonstrated by showing that $\partial^2 w^*/\partial t^2$ is just the negative of the third component of $\hat{\mathbf{u}}$ given by (2.4), and consequently equal by (1.10) to $-\left(\hat{w}^* + \frac{1}{\rho_0}\frac{\partial \hat{\varphi}}{\partial x_3} \right)$. Note that the $\delta(t)$ appears in the second differentiation of $\hat{w}^*$ due to the jump in $\partial \hat{w}^*/\partial t$ at $t=0$. The verification of the second equation of (2.1) is equivalent to showing that ρ_0 div $\hat{\mathbf{u}} = 0$ for $x \neq 0$. This is accomplished by a routine calculation on (2.4) using the same device as in the computation of $\hat{\mathbf{u}}$. The vanishing of ρ_0 div $\hat{\mathbf{u}}$ for $x \neq 0$ implies that the mass outflow $\hat{Q}(t)$ from the origin can be calculated from $\hat{Q}(t) = \rho_0 \int_S \hat{\mathbf{u}} \cdot \mathbf{n} ds$ for any fixed closed surface S that contains the origin. Taking S to be the sphere $|x| = 1$ we find $\hat{Q}(t) = \delta(t)$. The computation is aided by using polar coordinates, observing that $\int_0^\pi J_0(t\cos\theta)\sin\theta\, d\theta = \frac{2}{t}\int_0^t J_0(\tau)d\tau$, and using the relation[4] $[J_0(t)]^* [J_1(t)/t] = J_1(t)$. Notice that only the leading term in (2.4) gives a nonzero contribution to $\hat{Q}(t)$.

2.2. Representation of Solutions

The fundamental solution (2.3) provides a representation for solutions of the initial value problem (2.1) and (2.2) with only mild restrictions on μ, f, and g.

To begin we consider the case with f and g equal to zero. Then φ and w^* are given by the integral formulas

$$\left.\begin{aligned} \varphi(x,t) &= \int_0^t \int_{R^3} \hat{\varphi}(x-y, t-\tau)\mu(y,\tau)dyd\tau, \\ w^*(x,t) &= \int_0^t \int_{R^3} \hat{w}^*(x-y, t-\tau)\mu(y,\tau)dyd\tau. \end{aligned}\right\} \tag{2.5}$$

The parallel formula for the velocity is

$$\mathbf{u}(x,t) = \int_0^t \int_{R^3} \hat{\mathbf{u}}(x-y, t-\tau)\mu(y,\tau)dyd\tau. \tag{2.6}$$

The verification that the velocity derived from (2.5) by the formulas (1.7) and (1.10) is that given by (2.6) and that the equations (2.1) are satisfied by (2.5) is given in [3]. The arguments there

[4] See [9] p. 380.

are principally of the type used in [5] to demonstrate the differentiability properties of generalized domain potentials for elliptic partial differential equations. A suitable set of conditions to impose on μ are that it be piecewise continuous in t for $t \geq 0$ for all x and satisfy a uniform Lipschitz condition in x for all $t \geq 0$. These are sufficient for the proofs in [3], but certainly not necessary.

On the other hand it is comparatively easy to see directly that w^* given by (2.5) satisfies $w^*(x, 0^+) = \dfrac{\partial w^*}{\partial t}(x, 0^+) = 0$. In fact, the expression for $\hat{w}^*$ in (2.3) shows that for small positive t

$$\hat{w}^* = -\frac{1}{4\pi\rho_0}\frac{x_3}{|x|^3} t\,[1 + t^2 B(\cos\theta, t)],$$

where B is uniformly bounded in t and $\cos\theta$. This implies that $w^*(x, t)$ given by (2.5) is $0(t^2)$ provided only that $\mu(x, t)$ is not too rough, e.g., continuous and uniformly bounded, for small nonzero t. From this the required initial conditions follow immediately.

Now suppose $\mu \equiv 0$ but that the initial functions f and g for w^* and $\partial w^*/\partial t$ are nonvanishing. We require that $\partial f/\partial x_3$ and $\partial g/\partial x_3$ exist and satisfy the same sort of smoothness conditions given for μ earlier. Let $h(x, t)$ be defined by

$$h(x, t) = \rho_0[f(x)\cos t + g(x)\sin t].$$

Then a solution to (2.1) with (2.2) is given by

$$\left.\begin{aligned} \varphi(x, t) &= -\int_0^t \int_{R^3} \hat{\varphi}(x-y, t-\tau)\ \frac{\partial h}{\partial y_3}(y, \tau)\, dy\, d\tau, \\ w^*(x, t) &= -\int_0^t \int_{R^3} \hat{w}^*(x-y, t-\tau)\ \frac{\partial h}{\partial y_3}(y, \tau)\, dy\, d\tau + \frac{h(x, t)}{\rho_0}, \end{aligned}\right\} \tag{2.7}$$

and the corresponding velocity is

$$\mathbf{u}(x, t) = -\int_0^t \int_{R^3} \hat{\mathbf{u}}(x-y, t-\tau)\frac{\partial h}{\partial y_3}(y, \tau)\, dy\, d\tau + \frac{h(x, t)\mathbf{k}}{\rho_0}. \tag{2.8}$$

Since the integral expression in w^* given by (2.7) and its derivative with respect to t vanish at $t = 0$, the function w^* has the same initial conditions as h/ρ_0 as required. Further, since h satisfies $\partial^2 h/\partial t^2 + h = 0$, the functions ϕ and w^* satisfy the first of (2.1). The second is also satisfied with $\mu \equiv 0$ since the contribution from the integrals in the solution is exactly cancelled by the contribution from the nonintegral term in w^*.

By linearity the sum of the solution given by (2.5) and that of (2.7) satisfy the full inhomogeneous initial-value problem (2.1), (2.2); the corresponding velocities are the sum of (2.6) and (2.8). Thus a full solution of (2.1), (2.2) is given by

$$\left.\begin{aligned} \varphi(x, t) &= \int_0^t \int_{R^3} \hat{\varphi}(x-y, t-\tau)\left[\mu(y, \tau) - \frac{\partial h}{\partial y_3}(y, \tau)\right] dy\, d\tau, \\ w^*(x, t) &= \int_0^t \int_{R^3} \hat{w}^*(x-y, t-\tau)\left[\mu(y, \tau) - \frac{\partial h}{\partial y_3}(y, \tau)\right] dy\, d\tau + \frac{h(x, t)}{\rho_0}, \end{aligned}\right\} \tag{2.9}$$

and

$$\mathbf{u}(x, t) = \int_0^t \int_{R^3} \hat{\mathbf{u}}(x-y, t-\tau)\left[\mu(y, \tau) - \frac{\partial h}{\partial y_3}(y, \tau)\right] dy\, d\tau + \frac{h(x, t)\mathbf{k}}{\rho_0}. \tag{2.10}$$

3. Properties of the Solutions

Here we examine some of the properties of the solutions of the initial-value problem (2.1), (2.2). We begin by pointing out some of the more evident properties of the fundamental solution (2.3) and the fundamental velocities (2.4).

By inspection of the expressions for $\hat{\varphi}$ and $\hat{\mathbf{u}}$ it is clear that each of them breaks naturally into two parts; the first is exactly that of the fundamental solution in R^3 for an ideal fluid of constant density ρ_0, and the remaining part the required modifications which account for the effects of buoyancy. Thus the potential and velocities can be conveniently written as $\hat{\varphi} = \hat{\varphi}^\circ + \hat{\varphi}^*$ and $\hat{\mathbf{u}} = \hat{\mathbf{u}}^\circ + \hat{\mathbf{u}}^*$, where

$$\hat{\varphi}^*(x,t) = -\frac{1}{4\pi|x|}\left\{\frac{\partial}{\partial t} J_0(t\cos\theta) + [J_0(t\cos\theta)]^* \left[\frac{J_1(t)}{t}\right]\right\}$$

$$\hat{\mathbf{u}}^*(x,t) = \frac{x}{4\pi\rho_0|x|^3}\left\{\frac{\partial^2}{\partial t^2} t\,J_0(t\cos\theta) + \left[\frac{\partial}{\partial t} t\,J_0(t\cos\theta)\right] * \left[\frac{J_1(t)}{t}\right]\right\} \tag{3.1}$$

Examination of $\hat{\varphi}^*$ and $\hat{\mathbf{u}}^*$ shows that each of them is $0(t)$ at $t=0$, and consequently the influence of buoyancy takes some time to have an important effect; however that effect is long lasting, as we shall see more fully later. Since the quantities in braces in (3.1) are bounded functions of $\cos\theta$ for each t, it is clear that the velocity field decays in space like $1/|x|^2$, i.e., just as in the ideal fluid case.

As we have noted earlier, the fundamental velocity field $\hat{\mathbf{u}}$ has vanishing divergence except at $x=0$ and has a net mass outflow $\delta(t)$ from the origin, but the same is true of the ideal fluid part $\hat{\mathbf{u}}^\circ$. This means that the buoyancy induced velocity $\hat{\mathbf{u}}^*$ is also divergence free and, while singular at the origin, has zero net mass outflow.

The symmetry properties of the fundamental velocity $\hat{\mathbf{u}}$ are those to be expected: axial symmetry about the x_3 axis and odd symmetry about the origin. It is perhaps more surprising that $\hat{\mathbf{u}}$ is purely radial. Since it is, and is also proportional to $1/r^2$, the corresponding *fundamental displacements* are also purely radial and can be calculated easily.

We now turn our attention to those places where the fundamental solution and fundamental velocities have simple expressions. These are the x_3 axis and the plane $x_3=0$, i.e., $\theta=0$, π, and $\theta=\pi/2$. In fact, we find for $\theta=0$ and π

$$\left.\begin{aligned} &\hat{\varphi} = \frac{-1}{4\pi|x|}\delta(t),\quad \hat{w}^* = -\frac{1}{4\pi\rho_0}\frac{x_3}{4\pi|x|^3}\sin t,\\ &\hat{u}_1 = \hat{u}_2 = 0,\quad \hat{u}_3 = \frac{1}{4\pi\rho_0}\frac{x_3}{|x|^3}\{\delta(t) - \sin t\}, \end{aligned}\right\} \tag{3.2}$$

and for $\theta=\pi/2$

$$\left.\begin{aligned} &\hat{\varphi} = \frac{-1}{4\pi|x|}\left[\delta(t) + \int_0^t \frac{J_1(\tau)}{\tau}\,d\tau\right], \qquad \hat{w}^* = 0,\\ &\hat{u}_\alpha = \frac{1}{4\pi\rho_0}\frac{x_\alpha}{|x|^3}\left[\delta(t) + \int_0^t \frac{J_1(\tau)}{\tau}\,d\tau\right], \qquad \alpha = 1,2; \qquad \hat{u}_3 = 0. \end{aligned}\right\} \tag{3.3}$$

Thus, directly above and below the disturbance the vertical velocity is excited into a permanent oscillation at the buoyancy frequency, while in the plane the velocities $\hat{u}_\alpha$ oscillate about and tend to the values $\frac{1}{4\pi\rho_0}\frac{x_\alpha}{|x|^3}$. It is not really surprising that the fluid injected at $t=0$ should try to squeeze into the lamina $x_3=0$ from which it came.

Some of the properties of the fundamental solution that have just been noted are reflected immediately in the properties of the solutions (2.9) and (2.10).

The decomposition of $\hat{\varphi}$ and $\hat{\mathbf{u}}$ has its direct counterpart. In fact, we see that in general we may write the solution of (2.1) and (2.3) as $\varphi = \varphi^\circ + \varphi^*$, w^*, and $\mathbf{u} = \mathbf{u}^\circ + \mathbf{u}^*$, where

$$\left.\begin{aligned}
\varphi^\circ(x,t) &= \int_{R_3} \frac{-1}{4\pi|x-y|}\left\{ \mu(y,t) - \frac{\partial h}{\partial y_3}(y,t)\right\} dy,\\
\varphi^*(x,t) &= \int_0^t \int_{R_3} \hat{\varphi}^*(x-y,t-\tau)\left\{ \mu(y,\tau) - \frac{\partial h}{\partial y_3}(y,\tau)\right\} dy\, d\tau,\\
w^*(x,t) &= \int_0^t \int_{R_3} \hat{w}^*(x-y,t-\tau)\left\{ \mu(y,\tau) - \frac{\partial h}{\partial y_3}(y,\tau)\right\} dy\, d\tau + \frac{h(x,t)}{\rho_0},
\end{aligned}\right\} \tag{3.4}$$

and

$$\left.\begin{aligned}
\mathbf{u}^\circ(x,t) &= \int_{R_3} \frac{1}{4\pi\rho_0}\frac{x-y}{|x-y|^3}\left\{ \mu(y,t) - \frac{\partial h}{\partial y_3}(y,t)\right\} dy,\\
\mathbf{u}^*(x,t) &= \int_0^t \int_{R_3} \hat{\mathbf{u}}^*(x-y,t-\tau)\left\{ \mu(y,\tau) - \frac{\partial h}{\partial y_3}(y,\tau)\right\} dy\, d\tau + \frac{h(x,t)\mathbf{k}}{\rho_0}.
\end{aligned}\right\} \tag{3.5}$$

Clearly, the effect of a source distribution μ is two-fold. First there is an ideal fluid response given by the expressions for φ° and $\mathbf{u}^\circ$; then there is a slower long-lasting response due to buoyancy given by the expressions for φ^*, w^*, and $\mathbf{u}^*$. The latter depends, as the integrals show, on the history of μ.

Suppose that μ is a source distribution that vanishes outside a fixed bounded region G of space and has a total strength $M(t) \equiv \int_G \mu(x,t)dx$. By a standard line of argument[5] it is clear that at distances from G that are large compared to the diameter of G the source distribution has very nearly the same effect as a concentrated source of strength M located at the center of G. This leads naturally to an interest in a variety of concentrated source distributions, but discussion of this topic would lead too far afield; we shall pursue it elsewhere [4].

It is interesting that at large distances directly above and below G one is led by (3.2) to expect very small horizontal velocities and vertical velocities close to

$$u_3 = \frac{1}{4\pi\rho_0}\frac{x_3-y_3}{|x-y|^3}\left[M(t) - \int_0^t M(\tau)\sin(t-\tau)d\tau\right],$$

where y is the center of G. The second term in the brace is exactly the response of a simple oscillator having the buoyancy frequency and driven by a force $M(t)$.

The influence of the initial conditions through the source-like term $-\dfrac{\partial h}{\partial y_3}$ in (3.4) and (3.5) has the same decomposition into an instantaneous part and a slowly responding part. The terms in φ° and $\mathbf{u}^\circ$ as well as $h(x,t)/\rho_0$ in w^* and $\mathbf{u}^*$ ring at the buoyancy frequency, and the latter with no change in shape from that given by the initial functions f and g.

If f and g are smooth and vanish outside a bounded region G^*, $-\partial h/\partial y_3$ has zero total source strength. Since h vanishes outside G^*, the effect of such an initial disturbance dies off at large distances faster than that of a source distribution of nonzero total strength.

It is worthwhile to note that if f and g are independent of the vertical coordinate, then $\partial h/\partial y_3$ vanishes, so the total influence of the initial conditions is given by the term h/ρ_0 in w^* and u_3^*. This

[5] See, for example, [6] Chapter II, Article 6.

can be seen directly from (2.1) and (2.2) as well. In fact, it is clear that the second of (2.1) is satisfied with $\mu \equiv 0$ by $\varphi \equiv 0$ and w^* independent of x_3. Then the first of (2.1) and (2.2) imply

$$w^* = f(x)\ \cos t + g(x)\ \sin t = h(x, t)/\rho_0.$$

4. The Fundamental Displacements

This section is devoted to a brief examination of the displacements caused by the fundamental velocity field. The disturbance is taken to be at the origin, and for convenience it is scaled by the factor α so that the velocities are given by those of (2.4) multiplied by α.

Since the velocity field is radial, each fluid particle is displaced radially by the flow depending on its original radius r_0 before the disturbance, its polar angle θ, and time. In fact, the radial position r of a particle satisfies the ordinary differential equation $dr/dt = \hat{u}_r$, which takes the form

$$\frac{dr}{dt} = \frac{\alpha}{4\pi\rho_0} \cdot \frac{1}{r^2} \hat{F}(\cos\theta, t), \tag{4.1}$$

where $\hat{F}$ is given by

$$\hat{F}(\cos\theta, t) = \delta(t) + \frac{\partial^2}{\partial t^2} t J_0(t\cos\theta) + \left[\frac{\partial}{\partial t} t J_0(t\cos\theta)\right] * \left[\frac{J_1(t)}{t}\right]_{*}, \tag{4.2}$$

The equation (4.1) is easily integrated to obtain

$$r(r_0, \theta; t) = \left[r_0^3 + \frac{3\alpha}{4\pi\rho_0} F(\cos\theta, t)\right]^{1/3}, \tag{4.3}$$

in which the quantity F, defined by $F(\cos\theta, t) \equiv \int_0^t \hat{F}(\cos\theta, \tau)d\tau$, is given explicitly by

$$F(\cos\theta, t) = \frac{\partial}{\partial t} t J_0(t\cos\theta) + [t J_0(t\cos\theta)] * \left[\frac{J_1(t)}{t}\right] \tag{4.4}$$

From (4.4) simple expressions can be obtained for F on the x_3 axis and in the plane $x_3 = 0$. These are

$$F(\cos\theta, t) = \begin{cases} \cos t, & \theta = 2\,0, \pi, \\ J_0(t) + t\displaystyle\int_0^t \frac{J_1(\tau)}{\tau}\, d\tau, & 2\,\theta = \pi/2 \end{cases} \tag{4.5}$$

For ease in interpretation we take $\alpha = \dfrac{4\pi}{3}\rho_0$ so that the coefficient multiplying F in (4.3) is one. This is equivalent to an impulsive injection at $t = 0$ of just enough fluid to fill a sphere of unit radius. Since $F(\cos\theta, 0^+) = 1$, we have $r(0^+) = (r_0^3 + 1)^{1/3}$, so the first thing to happen is that the fluid pops out about the origin to accommodate the injected fluid. Subsequently, because $F(\cos\theta, t) = 1 + 0(t^2)$ for small positive t, the initial displacements begin to change very slowly.

We noted earlier that the velocity $\hat{\mathbf{u}}$ is singular at $x = 0$ but does not have a net in- or outflow there after $t = 0$. As a result, fluid is taken into the origin from some directions and given out in others. This is evidenced in the consequence of (4.5) that on the x_3 axis all the points that satisfy $r_0 \leq 1$ before the flow starts are drawn into the origin during the interval $\pi/2 \leq t \leq \pi$. Further

out along the x_3 axis displacements oscillate up and down without ever reaching the origin. It also follows from (4.5) that the displacements in the plane $x_3 = 0$ grow without bound.

Vertical displacements in a stratified fluid are of considerable interest, for they are often the easiest to measure. We note that they can be obtained equally well by interpreting (4.3). If we designate a particle in terms of its original height $z_0 = r_0 - \cos\theta$ and θ, the relation (4.3) also gives $z = r\cos\theta$ by

$$z(z_0, \theta; t) = \left[z_0^3 + \frac{3\alpha}{4\pi\rho_0} \cos^3\theta \, 2F(\cos\theta, t) \right]^{1/3}. \tag{4.6}$$

In a subsequent article [4] we shall give a more detailed analysis of the displacement and velocity fields of the fundamental and other singular solutions.

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Equivalence of Partitioned Matrices*

Robert B. Feinberg**

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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It is shown that if $M=\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is a partitioned matrix over a principal ideal domain R such that the matrices A and B are both square, then M is equivalent to $A \dot{+} B$ $\langle = \rangle$ the matric equation $T = AY + XB$ is solvable. The result is generalized to treat the case when

$$M=\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ 0 & M_{22} & \cdots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & \cdots & 0 & M_{tt} \end{bmatrix},$$

where each M_{ii} is square.

Key words: Determinantal divisors; equivalence; matric equation; partitioned matrix; Smith normal form.

Let R be a principal ideal domain and let R_{mn} denote the collection of $m \times n$ matrices over R. According to Theorem 2 of [2], if $A \epsilon R_{rr}$, $B \epsilon R_{ss}$, and $(\det A, \det B) = 1$, then for any $T \epsilon R_{rs}$,

$$S\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} = S\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $S(M)$ denotes the Smith normal form of a matrix M. The proof consists essentially of establishing two elementary propositions:

(i) For arbitrary $A \epsilon R_{rr}$, $B \epsilon R_{ss}$, and $T \epsilon R_{rs}$, if the matric equation (*) $T = AY + XB$ has a solution X, $Y \epsilon R_{rs}$, then

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $\tilde{E}$ denotes equivalence of matrices (ii). In the case when $(\det A, \det B) = 1$, (*) is always solvable.

The central result of this note (Theorem 1) provides a converse to (i), namely that if

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

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** Present address: Clarkson College of Technology, Potsdam, New York 13676.

then (*) must be solvable. We generalize this result to the case when

$$M=\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ 0 & M_{22} & \cdots & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & & M_{tt} \end{bmatrix},$$

where each M_{ii} is square, and also derive some corollaries.

Subsequent to completion of this work, the author discovered that Theorem 1 had been established in [3] in the case when R is the domain of polynomials over a field. The proof there carries over immediately to the case when R is an arbitrary P.I.D., and is similar to the proof of Theorem 1 presented here. The generalization of Theorem 1 is not developed there, however.

In the sequal R_n' will denote the group of unimodular $n \times n$ matrices over R, I_n will denote the identity matrix of order n, I will denote an identity matrix of unspecified order, 0_{mn} will denote the 0 matrix of order $m \times n$, 0_m will denote 0_{mm}, and $d_k[M]$ will denote the kth determinantal divisor of the matrix M.

See [1] for a good general reference on matrices over a P.I.D.

THEOREM 1: *Let* R *be a P.I.D.*, $A \epsilon R_{rr}$, $B \epsilon R_{ss}$, *and* $T \epsilon R_{rs}$. *Then* $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \langle = \rangle T = AY + XB$, *for suitable* $X, Y \epsilon R_{rs}$.

PROOF: ($\langle = $) Note that $\begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix}, \begin{bmatrix} I_r & Y \\ 0 & I_s \end{bmatrix} \epsilon R'_{r+s}$, and that

$$\begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_r & Y \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}.$$

Hence

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

($= \rangle$) Let $\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ be the statement we wish to prove, namely $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \langle = \rangle \exists X, Y \epsilon R_{r+s}$ such that $T = AY + XB$.

We will begin with four reduction steps

(i) We may assume w.l.o.g. (without loss of generality) that $A = S(A)$, $B = S(B)$. Justification: Choose U, $U^* \epsilon R'_{rr}$; V, $V^* \epsilon R'_{ss}$ such that $UAU^* = S(A)$, $VAV^* = S(B)$. Note that

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} S(A) & 0 \\ 0 & S(B) \end{bmatrix}$$

and that

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} = \begin{bmatrix} S(A) & \tilde{T} \\ 0 & S(B) \end{bmatrix},$$

where $\tilde{T} = UAV^*$.

Hence $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} S(A) & 0 \\ 0 & S(B) \end{bmatrix}$ and $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} S(A) & \tilde{T} \\ 0 & S(B) \end{bmatrix}$.

Thus $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} <=> \begin{bmatrix} S(A) & \hat{T} \\ 0 & S(B) \end{bmatrix} \tilde{E} \begin{bmatrix} S(A) & 0 \\ 0 & S(B) \end{bmatrix}$.

Note also that $T = AY + XB <=> UTV^* = UAYV^* + UXBV^* <=> T = UAU^* [(U^*)^{-1} YV^*] + UXV^{-1} [VBV^*] <=> \hat{T} = S(A)Y + X S(B)$, where $\hat{X} = UXV^{-1}$, $\hat{Y} = (U^*)^{-1} YV^*$. It follows that

$$\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} <=> \varphi \begin{bmatrix} S(A) & \hat{T} \\ 0 & S(B) \end{bmatrix}.$$

Hence setting $T = \hat{T}$, we may assume w.l.o.g. that $A = S(A)$, $B = S(B)$.

(ii) Let $r' = \text{rank } A$, $s' = \text{rank } B$. We may assume w.l.o.g. that $T = (t_{i,r+j})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$, where $t_{i,r+j} = 0$ for (i, j) such that $r' < i \leq r$ or $s' < i \leq s$. Justification: Let $A = S(A) = \text{diag}(\alpha_1, \ldots, \alpha_{r'}, 0, \ldots, 0)$, $B = S(B) = \text{diag}(\beta_1, \ldots, \beta_{s'}, 0, \ldots, 0)$, where $\alpha_1 | \alpha_2 | \ldots | \alpha_{r'}$ and $\beta_1 | \beta_2 | \ldots | \beta_{s'}$. Assume first that $r' < r$ and $s' < s$. If $\exists\, (i, j) \in (r', r] \times (s', s]$ such that $t_{i,r+j} \neq 0$, then it would follow that $\text{rank} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} > r' + s' = \text{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, a contradiction. Hence $t_{i,r+j} = 0$ for $(i, j) \in (r', r] \times (s', s]$.

Assume now that $r' < r$ and choose $(i, j) \in (r', r] \times [1, s']$. Then it is easily seen that $d_{r'+s'} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \alpha_1 \ldots \alpha_{r'} \beta_1 \ldots \beta_{s'}$, and that

$$\delta = \begin{cases} \alpha_1 \ldots \alpha_{r'} \beta_1 \ldots \beta_{j-1} \beta_{j+1} \ldots \beta_{s'} t_{i,r+j}, & j < s' \\ \alpha_1 \ldots \alpha_{r'} \beta_1 \ldots \beta_{j-1} t_{i,r+j}, & j = s' \end{cases}$$

is an $(r'+s') \times (r'+s')$ determinantal divisor of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. Since $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, it follows that $\alpha_1 \ldots \alpha_{r'} \beta_1 \ldots \beta_{s'} | \delta$, from which we deduce that $\beta_j | t_{i,r+j}$. Hence we may choose $w_{i,r+j} \in R$ such that $t_{i,r+j} = w_{i,r+j} \beta_j$.

Assume finally that $s' < s$ and choose $(i, j) \in [1, r'] \times (s', s]$. Then it is easily seen that

$$\eta = \begin{cases} \alpha_1 \ldots \alpha_{i-1} \alpha_{i+1} \ldots \alpha_{r'} \beta_1 \ldots \beta_{s'} t_{i,r+j}, & i < r' \\ \alpha_1 \ldots \alpha_{i-1} \beta_1 \ldots \beta_{s'} t_{i,r+j}, & i = r' \end{cases}$$

is an $(r'+s') \times (r'+s')$ determinantal divisor of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. It follows that $\alpha_1 \ldots \alpha_{r'} \beta_1 \ldots \beta_{s'} | \eta$, from which we deduce that $\alpha_i | t_{i,r+j}$. Hence we may choose $z_{i,r+j} \in R$ such that $t_{i,r+j} = \alpha_i z_{i,r+j}$.

Now for $1 \leq i \leq r$, $1 \leq j \leq s$ set

$$\bar{w}_{i,r+j} = \begin{cases} w_{i,r+j}, \text{ if } r' < i \leq r,\ 1 \leq j \leq s',\ t_{i,r+j} \neq 0 \\ 0 \quad \text{otherwise} \end{cases}$$

and set

$$\bar{z}_{i,r+j} = \begin{cases} z_{i,r+j} \text{ if } 1 \leq i \leq r',\ s' < j \leq s,\ t_{i,r+j} \neq 0 \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $\bar{W} = (\bar{w}_{i,r+j})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$, $Z = (\bar{z}_{i,r+j})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \in R_{r,s}$.

Then $\begin{bmatrix} I_r & -W \\ 0 & I_s \end{bmatrix} \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \begin{bmatrix} I_r & -Z \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix}$, where $\tilde{T} = T - AZ - WB$. Note that

$$(\tilde{T})_{i,r+j} = \begin{cases} T_{i,r+j}, 1 \leq i \leq r' \text{ or } 1 \leq j \leq s' \\ 0 \quad \text{otherwise.} \end{cases}$$

It follows that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix}$, so that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} <=> \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. It is also immediate that $T = AY + XB \ \langle = \rangle \ \tilde{T} = A\tilde{Y} + \tilde{X}B$, where $\tilde{X} = X - W$, $\tilde{Y} = Y - Z$. Thus $\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \langle = \rangle \ \varphi \begin{bmatrix} A & \tilde{T} \\ 0 & B \end{bmatrix}$. Here we may assume w.l.o.g. that $T = \tilde{T}$, i.e., that T is of the form specified above.

(iii) We may assume w.l.o.g. that rank $A = r$, rank $B = s$. Justification: We have from (i) and (ii) that we may assume that $A = \tilde{A} \dot{+} 0_{r-r'}$, where $\tilde{A} = \text{diag } (\alpha_1, \ldots, \alpha_{r'})$, $B = \tilde{B} \dot{+} 0_{s-s'}$, where $\tilde{B} = \text{diag } (\beta_1, \ldots, \beta_{s'})$, and that $T = \tilde{T} \dot{+} 0_{r-r', s-s'}$, where $\tilde{T} \in R_{r',s'}$. It is not difficult to show that

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \langle = \rangle \begin{bmatrix} \tilde{A} & \tilde{T} \\ 0 & \tilde{B} \end{bmatrix} \tilde{E} \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix},$$

also that $T = AY + XB$ for some $X, Y \in R_{rs} \langle = \rangle \tilde{T} = \tilde{A}\tilde{Y} + \tilde{X}\tilde{B}$ for some $\tilde{X}, \tilde{Y} \in R_{r's'}$. Thus

$$\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \langle = \rangle \varphi \begin{bmatrix} \tilde{A} & \tilde{T} \\ 0 & \tilde{B} \end{bmatrix}.$$

Hence we may assume w.l.o.g. that $A = \tilde{A}$, $B = \tilde{B}$, i.e., that rank $A = r$, *rank* $B = s$.

(iv) We may assume w.l.o.g. that $r = s$. Justification: Assume $r < s$. Let

$$\tilde{A} = I_{s-r} \dot{+} A, \ T = \begin{bmatrix} 0_{s-r,s} \\ T \end{bmatrix}.$$

It is an easy consequence of [1, Ch. 2, ex. 1] that

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \langle = \rangle \begin{bmatrix} \tilde{A} & \tilde{T} \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} \tilde{A} & 0 \\ 0 & B \end{bmatrix}.$$

It is also not difficult to show that $T = AY + XB$ for some $X, Y \in R_{rs} \langle = \rangle \tilde{T} = \tilde{A}\tilde{Y} + \tilde{X}B$ for some $\tilde{X}$, $\tilde{Y} \in R_{ss}$. Thus

$$\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \langle = \rangle \varphi \begin{bmatrix} \tilde{A} & \tilde{T} \\ 0 & B \end{bmatrix}.$$

Assume now that $s < r$. Let $\tilde{B} = I_{r-s} \dot{+} B$, $\tilde{T} = [0_{r,r-s}, T]$. Proceeding as above, we can show that

$$\varphi \begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \langle = \rangle \varphi \begin{bmatrix} A & \tilde{T} \\ 0 & \tilde{B} \end{bmatrix}.$$

It then follows from this and the above case that we may assume w.l.o.g. that $r = s$.

We now complete the proof of the theorem. By (i) – (iv), we may assume w.l.o.g. that $A = S(A) = \text{diag } (\alpha_1, \ldots, \alpha_r)$, $B = S(B) = \text{diag } (\beta_1, \ldots, \beta_r)$, where $\alpha_i, \beta_j \neq 0$, $1 \leq i, j \leq r$. Note that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \langle = \rangle \forall \ k \leq 2r, \ d_k \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Big|$ every $k \times k$ determinantal divisor of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. Note

also that $T=AY+XB \langle = \rangle (\alpha_i, \beta_j) \mid t_{i,r+j}$, $1 \leq i,j \leq r$. Now

$$d_k\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = (\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_k, \{\alpha_{r-t}\alpha_{r-t+1} \ldots \alpha_r\beta_1 \ldots \beta_{k-(t+1)}\}_{1 \leq t \leq k-2}),$$

as is easily calculated. We consider two cases.

a. $i \geq j$. Let $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i}$ denote the matrix obtained by deleting row $r+j$ and column i from $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. It is easily seen that if ρ is any $(i-1) \times (i-1)$ determinantal minor of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i}$, then $\rho t_{i,r+j}$ is an $i \times i$ determinantal minor of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. It then follows that

$$d_i\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Big| d_{i-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} t_{i,r+j}.$$

Note that

$$d_{i-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} = (\alpha_1 \ldots \alpha_{i-1}, \begin{cases} \beta_1 \ldots \beta_{j-1}\beta_{j+1} \ldots \beta_i, \text{if } j < i \\ \beta_1 \ldots \beta_{i-1}, \text{if } j = i \end{cases},$$

$$\{\alpha_1 \ldots \alpha_{i-u}\beta_1 \ldots \beta_{u-1}\}\, 2 \leq u \leq j,$$
$$\{\alpha_1 \ldots \alpha_{i-v}\beta_1 \ldots \beta_{j-1}\beta_{j+1} \ldots \beta_v\}_{j+1 \leq r \leq i-1}) \text{ (for } j \leq i-2).$$

From this it may be verified that

$$(\alpha_i, \beta_j)\, d_{i-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} \Big| d_i\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

It follows that $(a_i, \beta_j) \mid t_{i,r+j}$, as was to be shown.

b. $j > i$. We proceed as in (a). If σ is any $(j-1) \times (j-1)$ determinantal minor of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i}$ then $\sigma t_{i,r+j}$ is a $j \times j$ determinantal minor of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$. It then follows that

$$d_j\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Big| d_{j-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} t_{i,r+j}.$$

Note that

$$d_{j-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} = (\alpha_1 \ldots \alpha_{i-1}\alpha_{i+1} \ldots \alpha_j,$$

$$\beta_1 \ldots \beta_{j-1}, \{\alpha_1 \ldots \alpha_u\beta_1 \ldots \beta_{j-(u+1)}\}_{1 \leq u \leq i-1},$$

$$\{\alpha_1 \ldots \alpha_{i-1}\alpha_{i+1} \ldots \alpha_v\beta_1 \ldots \beta_{j-v}\}_{i+1 \leq v \leq j-1}).$$
(assuming $i \leq j+2$)

From this it may be verified that $(\alpha_i,\beta_j)d_{j-1}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{r+j,i} \Big| d_j\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

It follows that $(\alpha_i,\beta_j) \mid t_{i,r+j}$, as was to be shown.

With this we have completed proof of the theorem. Q.E.D.

COROLLARY 1.1: *Let* A, B, *and* R *be as before, and suppose* P, $Q \epsilon R_{rs}$.

Then $\begin{bmatrix} A & Q-P \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Rightarrow \begin{bmatrix} A & P \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & Q \\ 0 & B \end{bmatrix}$.

PROOF: By Theorem 1, $Q-P=AY+XB$, for some $X, Y \in R_{rs}$. *Note that*

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & P \\ 0 & B \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & Q \\ 0 & B \end{bmatrix}. \text{ Hence } \begin{bmatrix} A & P \\ 0 & B \end{bmatrix} \tilde{E} \begin{bmatrix} A & Q \\ 0 & B \end{bmatrix} \qquad \text{Q.E.D.}$$

NOTE: The converse to Corollary 1.2 fails. For example, $\begin{bmatrix} 6 & 4 \\ 0 & 9 \end{bmatrix} \tilde{E} \begin{bmatrix} 6 & 2 \\ 0 & 9 \end{bmatrix}$, as may be verified by considering determinantal divisors, but $\begin{bmatrix} 6 & 4-2 \\ 0 & 9 \end{bmatrix}$ is *not* equivalent to $\begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}$.

We now generalize Theorem 1 as follows:

THEOREM 2: *Let* M *be a matrix over* R, *and suppose that* M *may be partitioned as*

$$\mathrm{M} = \begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \cdots \mathrm{M}_{1t} \\ 0 & \mathrm{M}_{22} \cdots \mathrm{M}_{2t} \\ \vdots & \vdots \\ 0 \cdots 0 & \mathrm{M}_{tt} \end{bmatrix}$$

where each M_{ii} *is square, of order* r_i. *Then* $\mathrm{M} \tilde{\mathrm{E}}$ *diag*$[\mathrm{M}_{11}, \ldots, \mathrm{M}_{tt}]$ $\langle = \rangle$ *for* $1 \leq i < j \leq t \ni \mathrm{X}_{ij}$, $\mathrm{Y}_{ij} \in \mathrm{R}_{r_i r_j}$ *such that* $\mathrm{M}_{ij} = \mathrm{M}_{ii}\mathrm{Y}_{ij} + \sum_{k=i+1}^{j} \mathrm{X}_{ik}\mathrm{M}_{kj}$.

PROOF ($\Leftarrow$): Let

$$U = \begin{bmatrix} 1 & X_{12} \cdots & X_{1t} \\ 0 & 1 \quad 0 \cdots & 0 \\ \vdots & & \vdots \\ \vdots & & 0 \\ 0 \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & X_{23} \cdots & X_{2t} \\ \vdots & & & \vdots \\ 0 \cdots & & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 \cdots & 0 \cdots & 0 \\ & & \vdots & \vdots \\ & & 1 & \mathrm{X}_{t-1,t} \\ 0 \cdots & & 0 & 1 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & 1 & \vdots \\ & & & Y_{t-1,t} \\ 0 \cdots & 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots & 0 & 0 \\ 0 & 1 & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \vdots & Y_{t-2,t-1} & Y_{t-2,t} & \\ & & 1 & 0 \\ 0 \cdots & 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & Y_{12} \cdots & Y_{1t} \\ 0 & 1 \quad 0 & 0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & 0 \\ 0 \cdots & 0 & 1 \end{bmatrix}$$

Then U, V are invertible and U diag$[M_{11}, \ldots, M_{tt}]$ $V = M$, as may be verified. Hence $M \tilde{E}$ diag $[M_{11}, \ldots, M_{tt}]$.

($\Rightarrow$) Let $A_1 = M_{11}$, $T_1 = [M_{12}, \ldots, M_{1t}]$ and

$$B_1 = \begin{bmatrix} M_{22} & \cdots & M_{2t} \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & M_{tt} \end{bmatrix}$$

so that $M = \begin{bmatrix} A_1 & T_1 \\ 0 & B_1 \end{bmatrix}$. Note that to obtain a minor of $\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}$ with nonzero determinant, it is necessary that the number of rows deleted which pass through the block A_1 equal the number of columns deleted which pass through this block. It follows from this that every determinantal divisor of $\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}$ is also a determinantal divisor of M. Since $M \tilde{E}$ diag$[M_{11}, \ldots, M_{tt}]$, it follows that $\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} \tilde{E}$ diag$[M_{11}, \ldots, M_{tt}]$ as well, so that $M \tilde{E} \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}$. Hence by Theorem 1 there are matrices X_1, Y_1 of appropriate size such that $T_1 = A_1 Y_1 + X_1 B$. We may write $X_1 \equiv [X_{12}, \ldots, X_{1t}]$, $Y_1 = [Y_{12}, \ldots, Y_{1t}]$, where $X_{1j}, Y_{1j} \epsilon R_{r_1, r_j}$, $2 \leq j \leq t$, from which it follows that $M_{1j} = M_{11} Y_{1j} + \sum_{k=2}^{j} X_{1k} M_{kj}$.

Now let $A_2 = M_{22}$, $T_2 = [M_{23}, \ldots, M_{2t}]$,

$$B_2 = \begin{bmatrix} M_{33} & \cdots & M_{3t} \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots\ 0 & M_{tt} \end{bmatrix}$$

Then $M\tilde{E}A_1 \dot{+} \begin{bmatrix} A_2 & T_2 \\ 0 & B_2 \end{bmatrix} \tilde{E} A_1 \dot{+} \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}$ as before, so that $\begin{bmatrix} A_2 & T_2 \\ 0 & B_2 \end{bmatrix} \tilde{E} \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}$. Proceeding as above we obtain that $\exists X_2 \equiv [X_{23}, \ldots, X_{2t}], Y_2 = [Y_{23}, \ldots, Y_{2t}]$ such that

$$M_{2j} = M_{22} Y_{2j} \dot{+} \sum_{k=3}^{j} X_{2k} M_{kj}, 3 \leq j \leq t.$$

Continuing in this manner we obtain the desired linear recurrence relation.

Q.E.D.

COROLLARY 2.1: *Let* R *and* $\{M_{ii}\}_{i=1}^{t}$ *be as in Theorem 2, and*

$$M \equiv \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ 0 & M_{22} & \cdots & M_{2t} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & M_{tt} \end{bmatrix}$$

Then $M\tilde{E}$ *diag*$[M_{11}, M_{22}, \ldots, M_{tt}] \forall \{M_{ij}\}_{1 \leq i < j \leq t}$, *where* $M_{ij} \epsilon R_{r_i r_j}$ $\langle = \rangle$

(i) $(det\ M_{ii}, det\ M_{jj}) = 1$, $1 \leq i \leq j \leq t$ *or*

(ii) $\exists j \epsilon [1, n]$ *such that* $det\ M_{jj} = 0$ *and such that for all* $k \neq j$, M_{kk} *is unimodular.*

PROOF: ($\langle=$) (i) This is essentially Theorem 3 of [1]. (ii) Consider first the equation

$$M_{t-1,t}=M_{t-1,t-1}Y_{t-1,t}+X_{t-1,t}M_{tt}. \tag{*}$$

By hypothesis, at least one of $M_{t-1,t-1}$, M_{tt} is unimodular. If $M_{t-1,t-1}$ is unimodular, we may let $X_{t-1,t}=0$ and $Y_{t-1,t}=M_{t-1,t-1}^{-1}M_{t-1,t}$ to obtain a solution to (*). If M_{tt} is unimodular, it is again easy to solve (*).

Consider now the equations

$$M_{t-2,t-1}=M_{t-2,t-2}Y_{t-2,t-1}+X_{t-2,t-1}M_{t-1,t-1} \tag{**}$$

and

$$M_{t-2,t}=M_{t-2,t-2}Y_{t-2,t}+X_{t-2,t-1}M_{t-1,t}+X_{t-2,t}M_{tt} \tag{***}$$

Proceeding as above, it is again easy to solve (**), this time for $X_{t-2,t-1}$, $Y_{t-2,t-1}$. Now rewrite (***) as

$$M_{t-2,t}-X_{t-2,t-1}M_{t-1,t}=M_{t-2,t-2}Y_{t-2,t}+X_{t-2,t}M_{tt} \tag{***$'$}$$

and note that the matrices on the left-hand side have all been given or determined previously. Again, it is easy to solve (***$'$), for $X_{t-2,t}$, $Y_{t-2,t}$.

Proceeding in this manner, for $1\leq i<j\leq t$ we may find X_{ij}, $Y_{ij}\epsilon R_{r_i r_j}$ such that $M_{ij}=M_{ij}Y_{ij}+\sum_{k=i+1}^{j}X_{ik}M_{kj}$. Hence by Theorem 2, $M\tilde{E}$ diag$[M_{11}, M_{22}, \ldots, M_{tt}]$.

($=\rangle$) We may assume w.o.l.g. that $M_{ii}=S(M_{ii})=\text{diag}(\alpha_{i1}, \ldots, \alpha_{ir_i}, 0, \ldots, 0)$, $1\leq i\leq t$, where $r_{i'}\leq r_i$. Suppose $\exists$ distinct i, $j\epsilon[1, n]$ such that det $M_{ii}=\det M_{jj}=0$. Then $r_i'<r_i$ and $r'<r_j$. But if we let M_{ij} be the matrix of all 1's and set $M_{uv}=0$, $u\neq i$ or $v\neq j$, we would obtain that rank $M>$ rank diag$[M_{11}, \ldots, M_{tt}]$, contradicting hypothesis. Hence there is at most one $i\epsilon[1,n]$ such that det $M_{ii}=0$.

Suppose first that there is such an i. We will show that (i) holds in this case. Choose any $j>i$ (if such exist) and let M_{ij} and M_{uv} be defined as above. By Theorem 2,

$$M_{ij}=M_{ii}Y_{ij}+\sum_{k=i+1}^{j}X_{ik}M_{kj}=M_{ii}Y_{ij}+X_{ij}M_{jj}.$$

Considering the (k, l) component of this matric equation, for $r_{i'}<k\leq r_i$ and $l\leq r_j$ we obtain that $1=(x_{ij})_{kl}\alpha_{jl}$. This implies that α_{jl} is a unit, $1\leq l\leq r_j$. Since det $M_{jj}=\prod_{i=1}^{rj}\alpha_{jl}$, we obtain that M_{jj} is unimodular. If we choose any $j<i$, we may obtain by a similar argument that det M_{jj} is a unit, so that M_{jj} is unimodular in this case as well. We can thus conclude that (i) holds when det $M_{ii}=0$ for some i.

Now suppose that det $M_{ii}\neq 0$, $1\leq i\leq t$. We will show that (ii) holds in this case. Choose $i\neq j\epsilon[1, t]$ and let M_{ij} and $M_{uv}(u\neq i$ or $v\neq j)$ be defined as before. Again, $M_{ij}=M_{ii}Y_{ij}+X_{ij}M_{jj}$. Considering the (k, l) component of this matric equation, for $1\leq k\leq r_i$, $1\leq l\leq r_j$ we obtain that

$$1=\alpha_{ij}(Y_{ij})_{kl}+(X_{ij})_{kl}\alpha_{jl}.$$

It follows that $(\alpha_{ik}, \alpha_{jl}) = 1$. Since $\det M_{ii} = \prod_{k=1}^{r_i} \alpha_{ik}$ and $\det M_{jj} = \prod_{l=1}^{r_j} \alpha_{jl}$, we obtain that $(\det M_{ii}, \det M_{ij}) = 1$. We can thus conclude that (ii) holds when $\det M_{ii} \neq 0$, all i.

This completes the proof of the corollary.

Q.E.D.

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Tables of Binomial Coefficients and Sterling Numbers

Karl Goldberg, Frank Thomson Leighton, Morris Newman, and Susan Lana Zuckerman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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Tables of Binomial coefficients and of Stirling numbers are given, along with the most important formulas and relationships satisfied by them. The tables were computed using modular arithmetic.

Key words: Binomial coefficients; modular computation; Stirling numbers.

The binomial coefficients $\binom{n}{m}$ with m and n integers, $n \geq m \geq 0$, are defined by

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

Equivalently

$$\binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)}{m!}.$$

For any complex number x the general binomial coefficient is defined by

$$\binom{x}{m} = \frac{x(x-1)\ldots(x-m+1)}{m!} \qquad m=0,1,\ldots.$$

By convention $0!=1$ and $\binom{x}{0}=1$. Also $x(x-1)\ldots(x-m+1)\equiv 1$ for $m=0$.

The Stirling **I** numbers $\mathbf{I}_n^{(m)}$ appear when we expand $\binom{x}{n}$ in powers of x:

$$(1) \qquad n!\binom{x}{n} = x(x-1)\ldots(x-n+1) = \sum_{m=0}^{n} (-1)^{n-m}\mathbf{I}_n^{(m)}x^m \qquad n=0,1,\ldots.$$

For example:

$$x(x-1)(x-2)(x-3) = x^4-6x^3+11x^2-6x.$$

The Stirling **II** numbers $\mathbf{II}_n^{(m)}$ appear when we reverse equation (1) and express the powers of x in terms of the binomial coefficients:

$$(2) \qquad x^n = \sum_{m=0}^{n} m!\,\mathbf{II}_n^{(m)}\binom{x}{m} = \sum_{m=0}^{n} \mathbf{II}_n^{(m)}x(x-1)\ldots(x-m+1) \qquad n=0,1,\ldots.$$

For example:

$$x^4 = x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x.$$

AMS Subject Classification: 1004, 10A40, 65A05.

Equations (1) and (2) define the Stirling numbers uniquely. Special cases are:

(3)
$$\mathbf{I}_n^{(0)} = \mathbf{II}_n^{(0)} = \delta_{0n} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1, 2, \ldots \end{cases}$$

$$\mathbf{I}_n^{(1)} = (n-1)!,\ \mathbf{II}_n^{(1)} = 1 \qquad n = 1, 2, \ldots$$

(4)
$$\mathbf{I}_n^{(n)} = \mathbf{II}_n^{(n)} = 1 \qquad n = 0, 1, \ldots$$

$$\mathbf{I}_n^{(n-1)} = \mathbf{II}_n^{(n-1)} = \binom{n}{2} \qquad n = 1, 2, \ldots$$

$$\mathbf{I}_n^{(n-2)} = 3\binom{n}{4} + 2\binom{n}{3} \qquad n = 2, 3, \ldots$$

$$\mathbf{II}_n^{(n-2)} = 3\binom{n}{4} + \binom{n}{3} \qquad n = 2, 3, \ldots$$

The last three lines suggest that we can write $\mathbf{I}_n^{(n-m)}$ and $\mathbf{II}_n^{(n-m)}$ as sums of m binomial coefficients, and this is true. The coefficients are the associated Stirling numbers. (Sometimes called Jordan numbers or Ward numbers.)

$$\mathbf{I}_n^{(n-m)} = \sum_{k=0}^{m} \mathbf{I}_m'^{(k)} \binom{n}{m+k} \qquad n \geq m \geq 0$$

$$\mathbf{II}_n^{(n-m)} = \sum_{k=0}^{m} \mathbf{II}_m'^{(k)} \binom{n}{m+k} \qquad n \geq m \geq 0$$

Special cases are:

(5)
$$\mathbf{I}_m'^{(0)} = \mathbf{II}_m'^{(0)} = \delta_{0m} \qquad m = 0.\ 1, \ldots$$

$$\mathbf{I}_m'^{(1)} = m!,\ \mathbf{II}_m'^{(1)} = 1 \qquad m = 1, 2, \ldots$$

(6)
$$\mathbf{I}_m'^{(m)} = \mathbf{II}_m'^{(m)} = 1 \cdot 3 \cdot 5 \ldots (2m-1) \qquad m = 1, 2, \ldots$$
$$= \frac{(2m)!}{2^m \cdot m!} \qquad m = 0, 1, \ldots$$

The binomial coefficients can be computed by using the "Pascal's Triangle" recursion:

(7)
$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1} \qquad n \geq m \geq 1$$

with

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad n = 0, 1, \ldots$$

The Stirling numbers have similar recursions:

(8)
$$\mathbf{I}_{n+1}^{(m)} = n\,\mathbf{I}_n^{(m)} + \mathbf{I}_n^{(m-1)} \qquad n \geq m \geq 1$$

(9)
$$\mathbf{II}_{n+1}^{(m)} = m\,\mathbf{II}_n^{(m)} + \mathbf{II}_n^{(m-1)} \qquad n \geq m \geq 1$$

(10)
$$\mathbf{I}_{n+1}'^{(m)} = (n+m)\ (\mathbf{I}_n'^{(m)} + \mathbf{I}_n'^{(m-1)}) \qquad n \geq m \geq 1$$

(11) $$\mathbf{II}_{n+1}^{\prime(m)} = m\mathbf{II}_n^{\prime(m)} + (n+m)\mathbf{II}_n^{\prime(m-1)} \qquad n \geq m \geq 1$$

with eqs (3), (4), (5), and (6), yielding the end points.

The natural generating functions for the binomial coefficients are

$$(1+z)^n = \sum_{m=0}^{n} \binom{n}{m} z^m \qquad n = 0, 1, \ldots$$

and

$$\frac{z^m}{(1-z)^{m+1}} = \sum_{n=m}^{\infty} \binom{n}{m} z^n \qquad m = 0, 1, \ldots.$$

For the Stirling numbers we use the functions

$$\log\,(1-z)^{-1} = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

and

$$e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots$$

(12) $$\{\log\,(1-z)^{-1}\}^m = m! \sum_{n=m}^{\infty} \mathbf{I}_n^{(m)} \frac{z^n}{n!} \qquad m = 0, 1, \ldots$$

(13) $$\{e^z - 1\}^m = m! \sum_{n=m}^{\infty} \mathbf{II}_n^{(m)} \frac{z^n}{n!} \qquad m = 0, 1, \ldots$$

$$\{\log\,(1-z)^{-1} - z\}^m = m! \sum_{n=m}^{\infty} \mathbf{I}_n^{\prime(m)} \frac{z^{n+m}}{(n+m)!}$$

$$\{e^z - 1 - z\}^m = m! \sum_{n=m}^{\infty} \mathbf{II}_n^{\prime(m)} \frac{z^{n+m}}{(n+m)!} \qquad m = 0, 1, \ldots.$$

Setting $m = 1$ we can check that $\mathbf{I}_n^{(1)} = (n-1)!$, $\mathbf{II}_n^{(1)} = 1$, $\mathbf{I}_n^{\prime(1)} = n!$, and $\mathbf{II}_n^{\prime(1)} = 1$ for $n = 1, 2, \ldots$.

Substituting $-z = e^y - 1$ in eq (12), and $z = \log\,(1+y)$ in eq (13), we see that the Stirling **I** and Stirling **II** numbers are "orthogonal" to each other:

$$\sum_{k=m}^{n} (-1)^{k-m} \mathbf{I}_k^{(m)} \mathbf{II}_n^{(k)} = \delta_{mn} \qquad n \geq m \geq 0$$

and

$$\sum_{k=m}^{n} (-1)^{k-m} \mathbf{I}_n^{(k)} \mathbf{II}_k^{(m)} = \delta_{mn} \qquad n \geq m \geq 0$$

(δ_{mn} equals 1 if $m = n$ and is otherwise equal to 0.) This is the reason for the "inverse pair" of equations involving powers and factorials (eqs (1) and (2).); one implies the other.

Among the Stirling numbers only the Stirling **II** numbers have a simple closed form:

$$m!\,\mathbf{II}_n^{(m)} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n \qquad m, n \geq 0.$$

For $m > n \geq 0$ the sum equals 0.

The binomial coefficients and the Stirling **I** and Stirling **II** numbers have combinatorial significance. Let $S_n=\{1, 2, \ldots, n\}$ denote a typical set of order n. Given an integer m, $n \geq m \geq 0$, we can choose a subset of order m, or we can permute the symbols of S_n in such a way that there are m cycles in the permutation, or we can partition S_n into m nonempty subsets. The number of ways we can do these things are $\binom{n}{m}$, $\mathbf{I}_n^{(m)}$ and $\mathbf{II}_n^{(m)}$ respectively.

As examples let $n=4$ and $m=2$. The *subsets* of order 2 of S_4 are

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \text{ and } \{3, 4\}.$$

There are $\binom{4}{2}=6$ such subsets. The *permutations* of S_4 with 2 cycles each are

$$\begin{array}{ll} & (1\ \ 2\ \ 3)(4),\ (1\ \ 3\ \ 2)(4),\ (1\ \ 2\ \ 4)(3),\ (1\ \ 4\ \ 2)(3) \\ & (1\ \ 3\ \ 4)(2),\ (1\ \ 4\ \ 3)(2),\ (2\ \ 3\ \ 4)(1),\ (2\ \ 4\ \ 3)(1) \\ (14) & (12)(34),\ (13)(24),\ \text{and } (14)(23). \end{array}$$

There are $\mathbf{I}_4^{(2)}=11$ such permutations. The *partitions* of S_4 into 2 subsets each are

$$\begin{array}{ll} & (\{1, 2, 3\}, \{4\}),\ (\{1, 2, 4\}, \{3\}),\ (\{1, 3, 4\}, \{2\}),\ (\{2, 3, 4\}, \{1\}) \\ (15) & (\{1, 2\}, \{3, 4\}),\ (\{1, 3\}, \{2, 4\}),\ (\{1, 4\}, \{2, 3\}). \end{array}$$

There are $\mathbf{II}_4^{(2)}=7$ such partitions.

A derangement is a permutation which leaves no element fixed. The number of derangements of S_n with m cycles each is $\mathbf{I}'^{(m)}_{n-m}$. For $n=4$, $m=2$ the derangements are the $\mathbf{I}'^{(2)}_2=3$ listed in line (14).

Finally, $\mathbf{II}'^{(m)}_{n-m}$ is the number of partitions of S_n into m subsets each of which has at least two elements. For $n=4$, $m=2$ we have $\mathbf{II}'^{(2)}_2=3$: the three partitions listed in line (15).

The quantities $(-1)^{n-m}\mathbf{I}_n^{(m)}$ and $\mathbf{II}_n^{(m)}$ are usually called Stirling numbers of the first and second kind respectively. In "An Introduction to Combinatorial Analysis" by John Riordan their respective notations are $s(n, m)$ and $S(n, m)$. Also in that book (pages 75 and 76) $\mathbf{I}'^{(m)}_n=d(n+m, m)$ and $\mathbf{II}'^{(m)}_n=b(n+m, m)$ are called associated Stirling numbers of the first and second kind respectively. The notations for the Stirling numbers in this paper are due to Karl Goldberg.

The numbers were first computed by means of the recurrence formulas (7)—(11), using "modular arithmetic". That is, they were first computed modulo p for several large primes p, and the results so obtained were combined by means of the Chinese remainder theorem to produce the exact answers. They were then recomputed using multiprecision arithmetic, and the results were compared. The results agreed in every detail.

Table of Binomial Coefficients $\binom{n}{m}$ $n=1(1)100$

Binomial Coefficients $n=1(1)22$

m	$\binom{1}{m}$	$\binom{2}{m}$	$\binom{3}{m}$	$\binom{4}{m}$	$\binom{5}{m}$	$\binom{6}{m}$	$\binom{7}{m}$	$\binom{8}{m}$	$\binom{9}{m}$	$\binom{10}{m}$	$\binom{11}{m}$	$\binom{12}{m}$	$\binom{13}{m}$	$\binom{14}{m}$
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		2	3	4	5	6	7	8	9	10	11	12	13	14
2				6	10	15	21	28	36	45	55	66	78	91
3						20	35	56	84	120	165	220	286	364
4								70	126	210	330	495	715	1001
5										252	462	792	1287	2002
6												924	1716	3003
7														3432

m	$\binom{15}{m}$	$\binom{16}{m}$	$\binom{17}{m}$	$\binom{18}{m}$	$\binom{19}{m}$	$\binom{20}{m}$	$\binom{21}{m}$	$\binom{22}{m}$
0	1	1	1	1	1	1	1	1
1	15	16	17	18	19	20	21	22
2	105	120	136	153	171	190	210	231
3	455	560	680	816	969	1140	1330	1540
4	1365	1820	2380	3060	3876	4845	5985	7315
5	3003	4368	6188	8568	11628	15504	20349	26334
6	5005	8008	12376	18564	27132	38760	54264	74613
7	6435	11440	19448	31824	50388	77520	1 16280	1 70544
8		12870	24310	43758	75582	1 25970	2 03490	3 19770
9				48620	92378	1 67960	2 93930	4 97420
10						1 84756	3 52716	6 46646
11								7 05432

m	$\binom{23}{m}$	$\binom{24}{m}$	$\binom{25}{m}$	$\binom{26}{m}$	$\binom{27}{m}$	$\binom{28}{m}$
0	1	1	1	1	1	1
1	23	24	25	26	27	28
2	253	276	300	325	351	378
3	1771	2024	2300	2600	2925	3276
4	8855	10626	12650	14950	17550	20475
5	33649	42504	53130	65780	80730	98280
6	1 00947	1 34596	1 77100	2 30230	2 96010	3 76740
7	2 45157	3 46104	4 80700	6 57800	8 88030	11 84040
8	4 90314	7 35471	10 81575	15 62275	22 20075	31 08105
9	8 17190	13 07504	20 42975	31 24550	46 86825	69 06900
10	11 44066	19 61256	32 68760	53 11735	84 36285	131 23110
11	13 52078	24 96144	44 57400	77 26160	130 37895	214 74180
12		27 04156	52 00300	96 57700	173 83860	304 21755
13				104 00600	200 58300	374 42160
14						401 16600

m	$\binom{29}{m}$	$\binom{30}{m}$	$\binom{31}{m}$	$\binom{32}{m}$	$\binom{33}{m}$
0	1	1	1	1	1
1	29	30	31	32	33
2	406	435	465	496	528
3	3654	4060	4495	4960	5456
4	23751	27405	31465	35960	40920
5	1 18755	1 42506	1 69911	2 01376	2 37336
6	4 75020	5 93775	7 36281	9 06192	11 07568
7	15 60780	20 35800	26 29575	33 65856	42 72048
8	42 92145	58 52925	78 88725	105 18300	138 84156
9	100 15005	143 07150	201 60075	280 48800	385 67100
10	200 30010	300 45015	443 52165	645 12240	925 61040
11	345 97290	546 27300	846 72315	1290 24480	1935 36720
12	518 95935	864 93225	1411 20525	2257 92840	3548 17320
13	678 63915	1197 59850	2062 53075	3473 73600	5731 66440
14	775 58760	1454 22675	2651 82525	4714 35600	8188 09200
15		1551 17520	3005 40195	5657 22720	10371 58320
16				6010 80390	11668 03110

m	$\binom{34}{m}$	$\binom{35}{m}$	$\binom{36}{m}$	$\binom{37}{m}$	$\binom{38}{m}$
0	1	1	1	1	1
1	34	35	36	37	38
2	561	595	630	666	703
3	5984	6545	7140	7770	8436
4	46376	52360	58905	66045	73815
5	2 78256	3 24632	3 76992	4 35897	5 01942
6	13 44904	16 23160	19 47792	23 24784	27 60681
7	53 79616	67 24520	83 47680	102 95472	126 20256
8	181 56204	235 35820	302 60340	386 08020	489 03492
9	524 51256	706 07460	941 43280	1244 03620	1630 11640
10	1311 28140	1835 79396	2541 86856	3483 30136	4727 33756
11	2860 97760	4172 25900	6008 05296	8549 92152	12033 22288
12	5483 54040	8344 51800	12516 77700	18524 82996	27074 75148
13	9279 83760	14763 37800	23107 89600	35624 67300	54149 50296
14	13919 75640	23199 59400	37962 97200	61070 86800	96695 54100
15	18559 67520	32479 43160	55679 02560	93641 99760	1 54712 86560
16	22039 61430	40599 28950	73078 72110	1 28757 74670	2 22399 74430
17	23336 06220	45375 67650	85974 96600	1 59053 68710	2 87811 43380
18			90751 35300	1 76726 31900	3 35780 00610
19					3 53452 63800

m	$\binom{39}{m}$	$\binom{40}{m}$	$\binom{41}{m}$	$\binom{42}{m}$
0	1	1	1	1
1	39	40	41	42
2	741	780	820	861
3	9139	9880	10660	11480
4	82251	91390	1 01270	1 11930
5	5 75757	6 58008	7 49398	8 50668
6	32 62623	38 38380	44 96388	52 45786
7	153 80937	186 43560	224 81940	269 78328
8	615 23748	769 04685	955 48245	1180 30185
9	2119 15132	2734 38880	3503 43565	4458 91810
10	6357 45396	8476 60528	11210 99408	14714 42973
11	16760 56044	23118 01440	31594 61968	42805 61376
12	39107 97436	55868 53480	78986 54920	1 10581 16888
13	81224 25444	1 20332 22880	1 76200 76360	2 55187 31280
14	1 50845 04396	2 32069 29840	3 52401 52720	5 28602 29080
15	2 51408 40660	4 02253 45056	6 34322 74896	9 86724 27616
16	3 77112 60990	6 28521 01650	10 30774 46706	16 65097 21602
17	5 10211 17810	8 87323 78800	15 15844 80450	25 46619 27156
18	6 23591 43990	11 33802 61800	20 21126 40600	35 36971 21050
19	6 89232 64410	13 12824 08400	24 46626 70200	44 67753 10800
20		13 78465 28820	26 91289 37220	51 37916 07420
21				53 82578 74440

Binomial Coefficients $n = 43(1)49$

m	$\binom{43}{m}$	$\binom{44}{m}$	$\binom{45}{m}$	$\binom{46}{m}$
0	1	1	1	1
1	43	44	45	46
2	903	946	990	1035
3	12341	13244	14190	15180
4	1 23410	1 35751	1 48995	1 63185
5	9 62598	10 86008	12 21759	13 70754
6	60 96454	70 59052	81 45060	93 66819
7	322 24114	383 20568	453 79620	535 24680
8	1450 08513	1772 32627	2155 53195	2609 32815
9	5639 21995	7089 30508	8861 63135	11017 16330
10	19173 34783	24812 56778	31901 87286	40763 50421
11	57520 04349	76693 39132	1 01505 95910	1 33407 83196
12	1 53386 78264	2 10906 82613	2 87600 21745	3 89106 17655
13	3 65768 48168	5 19155 26432	7 30062 09045	10 17662 30790
14	7 83789 60360	11 49558 08528	16 68713 34960	23 98775 44005
15	15 15326 56696	22 99116 17056	34 48674 25584	51 17387 60544
16	26 51821 49218	41 67148 05914	64 66264 22970	99 14938 48554
17	42 11716 48758	68 63537 97976	110 30686 03890	174 96950 26860
18	60 83590 48206	102 95306 96964	171 58844 94940	281 89530 98830
19	80 04724 31850	140 88314 80056	243 83621 77020	415 42466 71960
20	96 05669 18220	176 10393 50070	316 98708 30126	560 82330 07146
21	105 20494 81860	201 26164 00080	377 36557 50150	694 35265 80276
22		210 40989 63720	411 67153 63800	789 03711 13950
23				823 34307 27600

m	$\binom{47}{m}$	$\binom{48}{m}$	$\binom{49}{m}$
0	1	1	1
1	47	48	49
2	1081	1128	1176
3	16215	17296	18424
4	1 78365	1 94580	2 11876
5	15 33939	17 12304	19 06884
6	107 37573	122 71512	139 83816
7	628 91499	736 29072	859 00584
8	3144 57495	3773 48994	4509 78066
9	13626 49145	16771 06640	20544 55634
10	51780 66751	65407 15896	82178 22536
11	1 74171 33617	2 25952 00368	2 91359 16264
12	5 22514 00851	6 96685 34468	9 22637 34836
13	14 06768 48445	19 29282 49296	26 25967 83764
14	34 16437 74795	48 23206 23240	67 52488 72536
15	75 16163 04549	109 32600 79344	157 55807 02584
16	150 32326 09098	225 48489 13647	334 81089 92991
17	274 11888 75414	424 44214 84512	649 92703 98159
18	456 86481 25690	730 98370 01104	1155 42584 85616
19	697 31997 70790	1154 18478 96480	1885 16848 97584
20	976 24796 79106	1673 56794 49896	2827 75273 46376
21	1255 17595 87422	2231 42392 66528	3904 99187 16424
22	1483 38976 94226	2738 56572 81648	4969 98965 48176
23	1612 38018 41550	3095 76995 35776	5834 33568 17424
24		3224 76036 83100	6320 53032 18876

m	$\binom{50}{m}$	$\binom{51}{m}$	$\binom{52}{m}$
0	1	1	1
1	50	51	52
2	1225	1275	1326
3	19600	20825	22100
4	2 30300	2 49900	2 70725
5	21 18760	23 49060	25 98960
6	158 90700	180 09460	203 58520
7	998 84400	1157 75100	1337 84560
8	5368 78650	6367 63050	7525 38150
9	25054 33700	30423 12350	36790 75400
10	1 02722 78170	1 27777 11870	1 58200 24220
11	3 73537 38800	4 76260 16970	6 04037 28840
12	12 13996 51100	15 87533 89900	20 63794 06870
13	35 48605 18600	47 62601 69700	63 50135 59600
14	93 78456 56300	129 27061 74900	176 89663 44600
15	225 08295 75120	318 86752 31420	448 13814 06320
16	492 36896 95575	717 45192 70695	1036 31945 02115
17	984 73793 91150	1477 10690 86725	2194 55883 57420
18	1805 35288 83775	2790 09082 74925	4267 19773 61650
19	3040 59433 83200	4845 94722 66975	7636 03805 41900
20	4712 92122 43960	7753 51556 27160	12599 46278 94135
21	6732 74460 62800	11445 66583 06760	19199 18139 33920
22	8874 98152 64600	15607 72613 27400	27053 39196 34160
23	10804 32533 65600	19679 30686 30200	35287 03299 57600
24	12154 86600 36300	22959 19134 01900	42638 49820 32100
25	12641 06064 37752	24795 92664 74052	47755 11798 75952
26			49591 85329 48104

m	$\binom{53}{m}$	$\binom{54}{m}$	$\binom{55}{m}$
0	1	1	1
1	53	54	55
2	1378	1431	1485
3	23426	24804	26235
4	2 92825	3 16251	3 41055
5	28 69685	31 62510	34 78761
6	229 57480	258 27165	289 89675
7	1541 43080	1771 00560	2029 27725
8	8863 22710	10404 65790	12175 66350
9	44316 13550	53179 36260	63584 02050
10	1 94990 99620	2 39307 13170	2 92486 49430
11	7 62237 53060	9 57228 52680	11 96535 65850
12	26 67831 35710	34 30068 88770	43 87297 41450
13	84 13929 66470	110 81761 02180	145 11829 90950
14	240 39799 04200	324 53728 70670	435 35489 72850
15	625 03477 50920	865 43276 55120	1189 97005 25790
16	1484 45759 08435	2109 49236 59355	2974 92513 14475
17	3230 87828 59535	4715 33587 67970	6824 82824 27325
18	6461 75657 19070	9692 63485 78605	14407 97073 46575
19	11903 23579 03550	18364 99236 22620	28057 62722 01225
20	20235 50084 36035	32138 73663 39585	50503 72899 62205
21	31798 64418 28055	52034 14502 64090	84172 88166 03675
22	46252 57335 68080	78051 21753 96135	1 30085 36256 60225
23	62340 42495 91760	1 08592 99831 59840	1 86644 21585 55975
24	77925 53119 89700	1 40265 95615 81460	2 48858 95447 41300
25	90393 61619 08052	1 68319 14738 97752	3 08585 10354 79212
26	97346 97128 24056	1 87740 58747 32108	3 56059 73486 29860
27		1 94693 94256 48112	3 82434 53003 80220

Binomial Coefficients $n = 56(1)58$

m	$\binom{56}{m}$	$\binom{57}{m}$	$\binom{58}{m}$
0	1	1	1
1	56	57	58
2	1540	1596	1653
3	27720	29260	30856
4	3 67290	3 95010	4 24270
5	38 19816	41 87106	45 82116
6	324 68436	362 88252	404 75358
7	2319 17400	2643 85836	3006 74088
8	14204 94075	16524 11475	19167 97311
9	75759 68400	89964 62475	1 06488 73950
10	3 56070 51480	4 31830 19880	5 21794 82355
11	14 89022 15280	18 45092 66760	22 76922 86640
12	55 83833 07300	70 72855 22580	89 17947 89340
13	188 99127 32400	244 82960 39700	315 55815 62280
14	580 47319 63800	769 46446 96200	1014 29407 35900
15	1625 32494 98640	2205 79814 62440	2975 26261 58640
16	4164 89518 40265	5790 22013 38905	7996 01828 01345
17	9799 75337 41800	13964 64855 82065	19754 86869 20970
18	21232 79897 73900	31032 55235 15700	44997 20090 97765
19	42465 59795 47800	63698 39693 21700	94730 94928 37400
20	78561 35621 63430	1 21026 95417 11230	1 84725 35110 32930
21	1 34676 61065 65880	2 13237 96687 29310	3 34264 92104 40540
22	2 14258 24422 63900	3 48934 85488 29780	5 62172 82175 59090
23	3 16729 57842 16200	5 30987 82264 80100	8 79922 67753 09880
24	4 35503 17032 97275	7 52232 74875 13475	12 83220 57139 93575
25	5 57444 05802 20512	9 92947 22835 17787	17 45179 97710 31262
26	6 64644 83841 09072	12 22088 89643 29584	22 15036 12478 47371
27	7 38494 26490 10080	14 03139 10331 19152	26 25227 99974 48736
28	7 64869 06007 60440	15 03363 32497 70520	29 06502 42828 89672
29			30 06726 64995 41040

m	$\binom{59}{m}$	$\binom{60}{m}$	$\binom{61}{m}$
0	1	1	1
1	59	60	61
2	1711	1770	1830
3	32509	34220	35990
4	4 55126	4 87635	5 21855
5	50 06386	54 61512	59 49147
6	450 57474	500 63860	555 25372
7	3411 49446	3862 06920	4362 70780
8	22174 71399	25586 20845	29448 27765
9	1 25656 71261	1 47831 42660	1 73417 63505
10	6 28283 56305	7 53940 27566	9 01771 70226
11	27 98717 68995	34 27001 25300	41 80941 52866
12	111 94870 75980	139 93588 44975	174 20589 70275
13	404 73763 51620	516 68634 27600	656 62222 72575
14	1329 85222 98180	1734 58986 49800	2251 27620 77400
15	3989 55668 94540	5319 40891 92720	7053 99878 42520
16	10971 28089 59985	14960 83758 54525	20280 24650 47245
17	27750 88697 22315	38722 16786 82300	53683 00545 36825
18	64752 06960 18735	92502 95657 41050	1 31225 12444 23350
19	1 39728 15019 35165	2 04480 21979 53900	2 96983 17636 94950
20	2 79456 30038 70330	4 19184 45058 05495	6 23664 67037 59395
21	5 18990 27214 73470	7 98446 57253 43800	12 17631 02311 49295
22	8 96437 74279 99630	14 15428 01494 73100	22 13874 58748 16900
23	14 42095 49928 68970	23 38533 24208 68600	37 53961 25703 41700
24	21 63143 24893 03455	36 05238 74821 72425	59 43771 99030 41025
25	30 28400 54850 24837	51 91543 79743 28292	87 96782 54565 00717
26	39 60216 10188 78633	69 88616 65039 03470	121 80160 44782 31762
27	48 40264 12452 96107	88 00480 22641 74740	157 89096 87680 78210
28	55 31730 42803 38408	103 71994 55256 34515	191 72474 77898 09255
29	59 13229 07824 30712	114 44959 50627 69120	218 16954 05884 03635
30		118 26458 15648 61424	232 71417 66276 30544

Binomial Coefficients $n=62(1)64$

m	$\binom{62}{m}$	$\binom{63}{m}$	$\binom{64}{m}$
0	1	1	1
1	62	63	64
2	1891	1953	2016
3	37820	39711	41664
4	5 57845	5 95665	6 35376
5	64 71002	70 28847	76 24512
6	614 74519	679 45521	749 74368
7	4917 96152	5532 70671	6212 16192
8	33810 98545	38728 94697	44261 65368
9	2 02865 91270	2 36676 89815	2 75405 84512
10	10 75189 33731	12 78055 25001	15 14732 14816
11	50 82713 23092	61 57902 56823	74 35957 81824
12	216 01531 23141	266 84244 46233	328 42147 03056
13	830 82812 42850	1046 84343 65991	1313 68588 12224
14	2907 89843 49975	3738 72655 92825	4785 56999 58816
15	9305 27499 19920	12213 17342 69895	15951 89998 62720
16	27334 24528 89765	36639 52028 09685	48852 69370 79580
17	73963 25195 84070	1 01297 49724 73835	1 37937 01752 83520
18	1 84908 12989 60175	2 58871 38185 44245	3 60168 87910 18080
19	4 28208 30081 18300	6 13116 43070 78475	8 71987 81256 22720
20	9 20647 84674 54345	13 48856 14755 72645	19 61972 57826 51120
21	18 41295 69349 08690	27 61943 54023 63035	41 10799 68779 35680
22	34 31505 61059 66195	52 72801 30408 74885	80 34744 84432 37920
23	59 67835 84451 58600	93 99341 45511 24795	146 72142 75919 99680
24	96 97733 24733 82725	156 65569 09185 41325	250 64910 54696 66120
25	147 40554 53595 41742	244 38287 78329 24467	401 03856 87514 65792
26	209 76942 99347 32479	357 17497 52942 74221	601 55785 31271 98688
27	279 69257 32463 09972	489 46200 31810 42451	846 63697 84753 16672
28	349 61571 65578 87465	629 30828 98041 97437	1118 77029 29852 39888
29	409 89428 83782 12890	759 51000 49361 00355	1388 81829 47402 97792
30	450 88371 72160 34179	860 77800 55942 47069	1620 28801 05303 47424
31	465 42835 32552 61088	916 31207 04712 95267	1777 09007 60655 42336
32			1832 62414 09425 90534

m	$\binom{65}{m}$	$\binom{66}{m}$	$\binom{67}{m}$
0	1	1	1
1	65	66	67
2	2080	2145	2211
3	43680	45760	47905
4	6 77040	7 20720	7 66480
5	82 59888	89 36928	96 57648
6	825 98880	908 58768	997 95696
7	6961 90560	7787 89440	8696 48208
8	50473 81560	57435 72120	65223 61560
9	3 19667 49880	3 70141 31440	4 27577 03560
10	17 90137 99328	21 09805 49208	24 79946 80648
11	89 50689 96640	107 40827 95968	128 50633 45176
12	402 78104 84880	492 28794 81520	599 69622 77488
13	1642 10735 15280	2044 88840 00160	2537 17634 81680
14	6099 25587 71040	7741 36322 86320	9786 25162 86480
15	20737 46998 21536	26836 72585 92576	34578 08903 78896
16	64804 59369 42300	85542 06367 63836	1 12378 78953 56412
17	1 86789 71123 63100	2 51594 30493 05400	3 37136 36860 69236
18	4 98105 89663 01600	6 84895 60786 64700	9 36489 91279 70100
19	12 32156 69166 40800	17 30262 58829 42400	24 15158 19616 07100
20	28 33960 39082 73840	40 66117 08249 14640	57 96379 67078 57040
21	60 72772 26605 86800	89 06732 65688 60640	129 72849 73937 75280
22	121 45544 53211 73600	182 18316 79817 60400	271 25049 45506 21040
23	227 06887 60352 37600	348 52432 13564 11200	530 70748 93381 71600
24	397 37053 30616 65800	624 43940 90969 03400	972 96373 04533 14600
25	651 68767 42211 31912	1049 05820 72827 97712	1673 49761 63797 01112
26	1002 59642 18786 64480	1654 28409 60997 96392	2703 34230 33825 94104
27	1448 19483 16025 15360	2450 79125 34811 79840	4105 07534 95809 76232
28	1965 40727 14605 56560	3413 60210 30630 71920	5864 39335 65442 51760
29	2507 58858 77255 37680	4472 99585 91860 94240	7886 59796 22491 66160
30	3009 10630 52706 45216	5516 69489 29961 82896	9989 69075 21822 77136
31	3397 37808 65958 89760	6406 48439 18665 34976	11923 17928 48627 17872
32	3609 71421 70081 32870	7007 09230 36040 22630	13413 57669 54705 57606
33		7219 42843 40162 65740	14226 52073 76202 88370

Binomial Coefficients $n = 68(1)69$

m	$\binom{68}{m}$	$\binom{69}{m}$
0	1	1
1	68	69
2	2278	2346
3	50116	52394
4	8 14385	8 64501
5	104 24128	112 38513
6	1094 53344	1198 77472
7	9694 43904	10788 97248
8	73920 09768	83614 53672
9	4 92800 65120	5 66720 74888
10	29 07523 84208	34 00324 49328
11	153 30580 25824	182 38104 10032
12	728 20256 22664	881 50836 48488
13	3136 87257 59168	3865 07513 81832
14	12323 42797 68160	15460 30055 27328
15	44364 34071 65376	56687 76869 33536
16	1 46956 87862 35308	1 91321 21934 00684
17	4 49515 15814 25648	5 96472 03676 60956
18	12 73626 28140 39336	17 23141 43954 64984
19	33 51648 10895 77200	46 25274 39036 16536
20	82 11537 86694 64140	115 63185 97590 41340
21	187 69229 41016 32320	269 80767 27710 96460
22	400 97899 19443 96320	588 67128 60460 28640
23	801 95798 38887 92640	1202 93697 58331 88960
24	1503 67121 97914 86200	2305 62920 36802 78840
25	2646 46134 68330 15712	4150 13256 66245 01912
26	4376 83991 97622 95216	7023 30126 65953 10928
27	6808 41765 29635 70336	11185 25757 27258 65552
28	9969 46870 61252 27992	16777 88635 90887 98328
29	13750 99131 87934 17920	23720 46002 49186 45912
30	17876 28871 44314 43296	31627 28003 32248 61216
31	21912 87003 70449 95008	39789 15875 14764 38304
32	25336 75598 03332 75478	47249 62601 73782 70486
33	27640 09743 30908 45976	52976 85341 34241 21454
34	28453 04147 52405 76740	56093 13890 83314 22716

m	$\binom{70}{m}$	$\binom{71}{m}$
0	1	1
1	70	71
2	2415	2485
3	54740	57155
4	9 16895	9 71635
5	121 03014	130 19909
6	1311 15985	1432 18999
7	11987 74720	13298 90705
8	94403 50920	1 06391 25640
9	6 50335 28560	7 44738 79480
10	39 67045 24216	46 17380 52776
11	216 38428 59360	256 05473 83576
12	1063 88940 58520	1280 27369 17880
13	4746 58350 30320	5810 47290 88840
14	19325 37569 09160	24071 95919 39480
15	72148 06924 60864	91473 44493 70024
16	2 48008 98803 34220	3 20157 05727 95084
17	7 87793 25610 61640	10 35802 24413 95860
18	23 19613 47631 25940	31 07406 73241 87580
19	63 48415 82990 81520	86 68029 30622 07460
20	161 88460 36626 57876	225 36876 19617 39396
21	385 43953 25301 37800	547 32413 61927 95676
22	858 47895 88171 25100	1243 91849 13472 62900
23	1791 60826 18792 17600	2650 08722 06963 42700
24	3508 56617 95134 67800	5300 17444 13926 85400
25	6455 76177 03047 80752	9964 32794 98182 48552
26	11173 43383 32198 12840	17629 19560 35245 93592
27	18208 55883 93211 76480	29381 99267 25409 89320
28	27963 14393 18146 63880	46171 70277 11358 40360
29	40498 34638 40074 44240	68461 49031 58221 08120
30	55347 74005 81435 07128	95846 08644 21509 51368
31	71416 43878 47012 99520	1 26764 17884 28448 06648
32	87038 78476 88547 08790	1 58455 22355 35560 08310
33	1 00226 47943 08023 91940	1 87265 26419 96571 00730
34	1 09069 99232 17555 44170	2 09296 47175 25579 36110
35	1 12186 27781 66628 45432	2 21256 27013 84183 89602

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Binomial Coefficients $n = 72(1)73$

m	$\binom{72}{m}$	$\binom{73}{m}$
0	1	1
1	72	73
2	2556	2628
3	59640	62196
4	10 28790	10 88430
5	139 91544	150 20334
6	1562 38908	1702 30452
7	14731 09704	16293 48612
8	1 19690 16345	1 34421 26049
9	8 51130 05120	9 70820 21465
10	53 62119 32256	62 13249 37376
11	302 22854 36352	355 84973 68608
12	1536 32843 01456	1838 55697 37808
13	7090 74660 06720	8627 07503 08176
14	29882 43210 28320	36973 17870 35040
15	1 15545 40413 09504	1 45427 83623 37824
16	4 11630 50221 65108	5 27175 90634 74612
17	13 55959 30141 90944	17 67589 80363 56052
18	41 43208 97655 83440	54 99168 27797 74384
19	117 75436 03863 95040	159 18645 01519 78480
20	312 04905 50239 46856	429 80341 54103 41896
21	772 69289 81545 35072	1084 74195 31784 81928
22	1791 24262 75400 58576	2563 93552 56945 93648
23	3894 00571 20436 05600	5685 24833 95836 64176
24	7950 26166 20890 28100	11844 26737 41326 33700
25	15264 50239 12109 33952	23214 76405 32999 62052
26	27593 52355 33428 42144	42858 02594 45537 76096
27	47011 18827 60655 82912	74604 71182 94084 25056
28	75553 69544 36768 29680	1 22564 88371 97424 12592
29	1 14633 19308 69579 48480	1 90186 88853 06347 78160
30	1 64307 57675 79730 59488	2 78940 76984 49310 07968
31	2 22610 26528 49957 58016	3 86917 84204 29688 17504
32	2 85219 40239 64008 14958	5 07829 66768 13965 72974
33	3 45720 48775 32131 09040	6 30939 89014 96139 23998
34	3 96561 73595 22150 36840	7 42282 22370 54281 45880
35	4 30552 74189 09763 25712	8 27114 47784 31913 62552
36	4 42512 54027 68367 79204	8 73065 28216 78131 04916

m	$\binom{74}{m}$	$\binom{75}{m}$
0	1	1
1	74	75
2	2701	2775
3	64824	67525
4	11 50626	12 15450
5	161 08764	172 59390
6	1852 50786	2013 59550
7	17995 79064	19848 29850
8	1 50714 74661	1 68710 53725
9	11 05241 47514	12 55956 22175
10	71 84069 58841	82 89311 06355
11	417 98223 05984	489 82292 64825
12	2194 40671 06416	2612 38894 12400
13	10465 63200 45984	12660 03871 52400
14	45600 25373 43216	56065 88573 89200
15	1 82401 01493 72864	2 28001 26867 16080
16	6 72603 74258 12436	8 55004 75751 85300
17	22 94765 70998 30664	29 67369 45256 43100
18	72 66758 08161 30436	95 61523 79159 61100
19	214 17813 29317 52864	286 84571 37478 83300
20	588 98986 55623 20376	803 16799 84940 73240
21	1514 54536 85888 23824	2103 53523 41511 44200
22	3648 67747 88730 75576	5163 22284 74618 99400
23	8249 18386 52782 57824	11897 86134 41513 33400
24	17529 51571 37162 97876	25778 69957 89945 55700
25	35059 03142 74325 95752	52588 54714 11488 93628
26	66072 78999 78537 38148	1 01131 82142 52863 33900
27	1 17462 73777 39622 01152	1 83535 52777 18159 39300
28	1 97169 59554 91508 37648	3 14632 33332 31130 38800
29	3 12751 77225 03771 90752	5 09921 36779 95280 28400
30	4 69127 65837 55657 86128	7 81879 43062 59429 76880
31	6 65858 61188 78998 25472	11 34986 27026 34656 11600
32	8 94747 50972 43653 90478	15 60606 12161 22652 15950
33	11 38769 55783 10104 96972	20 33517 06755 53758 87450
34	13 73222 11385 50420 69878	25 11991 67168 60525 66850
35	15 69396 70154 86195 08432	29 42618 81540 36615 78310
36	17 00179 76001 10044 67468	32 69576 46155 96239 75900
37	17 46130 56433 56262 09832	34 46310 32434 66306 77300

Binomial Coefficients $n = 76(1)77$

m	$\binom{76}{m}$	$\binom{77}{m}$
0	1	1
1	76	77
2	2850	2926
3	70300	73150
4	12 82975	13 53275
5	184 74840	197 57815
6	2186 18940	2370 93780
7	21861 89400	24048 08340
8	1 88558 83575	2 10420 72975
9	14 24666 75900	16 13225 59475
10	95 45267 28530	109 69934 04430
11	572 71603 71180	668 16870 99710
12	3102 21186 77225	3674 92790 48405
13	15272 42765 64800	18374 63952 42025
14	68725 92445 41600	83998 35211 06400
15	2 84067 15441 05280	3 52793 07886 46880
16	10 83006 02619 01380	13 67073 18060 06660
17	38 22374 21008 28400	49 05380 23627 29780
18	125 28893 24416 04200	163 51267 45424 32600
19	382 46095 16638 44400	507 74988 41054 48600
20	1090 01371 22419 56540	1472 47466 39058 00940
21	2906 70323 26452 17440	3996 71694 48871 73980
22	7266 75808 16130 43600	10173 46131 42582 61040
23	17061 08419 16132 32800	24327 84227 32262 76400
24	37676 56092 31458 89100	54737 64511 47591 21900
25	78367 24672 01434 49328	1 16043 80764 32893 38428
26	1 53720 36856 64352 27528	2 32087 61528 65786 76856
27	2 84667 34919 71022 73200	4 38387 71776 35375 00728
28	4 98167 86109 49289 78100	7 82835 21029 20312 51300
29	8 24553 70112 26410 67200	13 22721 56221 75700 45300
30	12 91800 79842 54710 05280	21 16354 49954 81120 72480
31	19 16865 70088 94085 88480	32 08666 49931 48795 93760
32	26 95592 39187 57308 27550	46 12458 09276 51394 16030
33	35 94123 18916 76411 03400	62 89715 58104 33719 30950
34	45 45508 73924 14284 54300	81 39631 92840 90695 57700
35	54 54610 48708 97141 45160	100 00119 22633 11425 99460
36	62 12195 27696 32855 54210	116 66805 76405 29996 99370
37	67 15886 78590 62546 53200	129 28082 06286 95402 07410
38	68 92620 64869 32613 54600	136 08507 43459 95160 07800

m	$\binom{78}{m}$	$\binom{79}{m}$
0	1	1
1	78	79
2	3003	3081
3	76076	79079
4	14 26425	15 02501
5	211 11090	225 37515
6	2568 51595	2779 62685
7	26419 02120	28987 53715
8	2 34468 81315	2 60887 83435
9	18 23646 32450	20 58115 13765
10	125 83159 63905	144 06805 96355
11	777 86805 04140	903 69964 68045
12	4343 09661 48115	5120 96466 52255
13	22049 56742 90430	26392 66404 38545
14	1 02372 99163 48425	1 24422 55906 38855
15	4 36791 43097 53280	5 39164 42261 01705
16	17 19866 25946 53540	21 56657 69044 06820
17	62 72453 41687 36440	79 92319 67633 89980
18	212 56647 69051 62380	275 29101 10738 98820
19	671 26255 86478 81200	883 82903 55530 43580
20	1980 22454 80112 49540	2651 48710 66591 30740
21	5469 19160 87929 74920	7449 41615 68042 24460
22	14170 17825 91454 35020	19639 36986 79384 09940
23	34501 30358 74845 37440	48671 48184 66299 72460
24	79065 48738 79853 98300	1 13566 79097 54699 35740
25	1 70781 45275 80484 60328	2 49846 94014 60338 58628
26	3 48131 42292 98680 15284	5 18912 87568 79164 75612
27	6 70475 33305 01161 77584	10 18606 75597 99841 92868
28	12 21222 92805 55687 52028	18 91698 26110 56849 29612
29	21 05556 77250 96012 96600	33 26779 70056 51700 48628
30	34 39076 06176 56821 17780	55 44632 83427 52834 14380
31	53 25020 99886 29916 66240	87 64097 06062 86737 84020
32	78 21124 59208 00190 09790	131 46145 59094 30106 76030
33	109 02173 67380 85113 46980	187 23298 26588 85303 56770
34	144 29347 50945 24414 88650	253 31521 18326 09528 35630
35	181 39751 15474 02121 57160	325 69098 66419 26536 45810
36	216 66924 99038 41422 98830	398 06676 14512 43544 55990
37	245 94887 82692 25399 06780	462 61812 81730 66822 05610
38	265 36589 49746 90562 15210	511 31477 32439 15961 21990
39	272 17014 86919 90320 15600	537 53604 36666 80882 30810

Binomial Coefficients $n=80(1)81$

m	$\binom{80}{m}$	$\binom{81}{m}$
0	1	1
1	80	81
2	3160	3240
3	82160	85320
4	15 81580	16 63740
5	240 40016	256 21596
6	3005 00200	3245 40216
7	31767 16400	34772 16600
8	2 89875 37150	3 21642 53550
9	23 19002 97200	26 08878 34350
10	164 64921 10120	187 83924 07320
11	1047 76770 64400	1212 41691 74520
12	6024 66431 20300	7072 43201 84700
13	31513 62870 90800	37538 29302 11100
14	1 50815 22310 77400	1 82328 85181 68200
15	6 63586 98167 40560	8 14402 20478 17960
16	26 95822 11305 08525	33 59409 09472 49085
17	101 48977 36677 96800	128 44799 47983 05325
18	355 21420 78372 88800	456 70398 15050 85600
19	1159 12004 66269 42400	1514 33425 44642 31200
20	3535 31614 22121 74320	4694 43618 88391 16720
21	10100 90326 34633 55200	13636 21940 56755 29520
22	27088 78602 47426 34400	37189 68928 82059 89600
23	68310 85171 45683 82400	95399 63773 93110 16800
24	1 62238 27282 20999 08200	2 30549 12453 66682 90600
25	3 63413 73112 15037 94368	5 25652 00394 36037 02568
26	7 68759 81583 39503 34240	11 32173 54695 54541 28608
27	15 37519 63166 79006 68480	23 06279 44750 18510 02720
28	29 10305 01708 56691 22480	44 47824 64875 35697 90960
29	52 18477 96167 08549 78240	81 28782 97875 65241 00720
30	88 71412 53484 04534 63008	140 89890 49651 13084 41248
31	143 08729 89490 39571 98400	231 80142 42974 44106 61408
32	219 10242 65157 16844 60050	362 18972 54647 56416 58450
33	318 69443 85683 15410 32800	537 79686 50840 32254 92850
34	440 54819 44914 94831 92400	759 24263 30598 10242 25200
35	579 00619 84745 36064 81440	1019 55439 29660 30896 73840
36	723 75774 80931 70081 01800	1302 76394 65677 06145 83240
37	860 68488 96243 10366 61600	1584 44263 77174 80447 63400
38	973 93290 14169 82783 27600	1834 61779 10412 93149 89200
39	1048 85081 69105 96843 52800	2022 78371 83275 79626 80400
40	1075 07208 73333 61764 61620	2123 92290 42439 58608 14420

m	$\binom{82}{m}$	$\binom{83}{m}$
0	1	1
1	82	83
2	3321	3403
3	88560	91881
4	17 49060	18 37620
5	272 85336	290 34396
6	3501 61812	3774 47148
7	38017 56816	41519 18628
8	3 56414 70150	3 94432 26966
9	29 30520 87900	32 86935 58050
10	213 92802 41670	243 23323 29570
11	1400 25615 81840	1614 18418 23510
12	8284 84893 59220	9685 10509 41060
13	44610 72503 95800	52895 57397 55020
14	2 19867 14483 79300	2 64477 86987 75100
15	9 96731 05659 86160	12 16598 20143 65460
16	41 73811 29950 67045	51 70542 35610 53205
17	162 04208 57455 54410	203 78019 87406 21455
18	585 15197 63033 90925	747 19406 20489 45335
19	1971 03823 59693 16800	2556 19021 22727 07725
20	6208 77044 33033 47920	8179 80867 92726 64720
21	18330 65559 45146 46240	24539 42603 78179 94160
22	50825 90869 38815 19120	69156 56428 83961 65360
23	1 32589 32702 75170 06400	1 83415 23572 13985 25520
24	3 25948 76227 59793 07400	4 58538 08930 34963 13800
25	7 56201 12848 02719 93168	10 82149 89075 62513 00568
26	16 57825 55089 90578 31176	24 14026 67937 93298 24344
27	34 38452 99445 73051 31328	50 96278 54535 63629 62504
28	67 54104 09625 54207 93680	101 92557 09071 27259 25008
29	125 76607 62751 00938 91680	193 30711 72376 55146 85360
30	222 18673 47526 78325 41968	347 95281 10277 79264 33648
31	372 70032 92625 57191 02656	594 88706 40152 35516 44624
32	593 99114 97622 00523 19858	966 69147 90247 57714 22514
33	899 98659 05487 88671 51300	1493 97774 03109 89194 71158
34	1297 03949 81438 42497 18050	2197 02608 86926 31168 69350
35	1778 79702 60258 41138 99040	3075 83652 41696 83636 17090
36	2322 31833 95337 37042 57080	4101 11536 55595 78181 56120
37	2887 20658 42851 86593 46640	5209 52492 38189 23636 03720
38	3419 06042 87587 73597 52600	6306 26701 30439 60190 99240
39	3857 40150 93688 72776 69600	7276 46193 81276 46374 22200
40	4146 70662 25715 38234 94820	8004 10813 19404 11011 64420
41	4247 84580 84879 17216 28840	8394 55243 10594 55451 23660

m	$\binom{84}{m}$	$\binom{85}{m}$
0	1	1
1	84	85
2	3486	3570
3	95284	98770
4	19 29501	20 24785
5	308 72016	328 01517
6	4064 81544	4373 53560
7	45293 65776	49358 47320
8	4 35951 45594	4 81245 11370
9	36 81367 85016	41 17319 30610
10	276 10258 87620	312 91626 72636
11	1857 41741 53080	2133 52000 40700
12	11299 28927 64570	13156 70669 17650
13	62580 67906 96080	73879 96834 60650
14	3 17373 44385 30120	3 79954 12292 26200
15	14 81076 07131 40560	17 98449 51516 70680
16	63 87140 55754 18665	78 68216 62885 59225
17	255 48562 23016 74660	319 35702 78770 93325
18	950 97426 07895 66790	1206 45988 30912 41450
19	3303 38427 43216 53060	4254 35853 51112 19850
20	10735 99889 15453 72445	14039 38316 58670 25505
21	32719 23471 70906 58880	43455 23360 86360 31325
22	93695 99032 62141 59520	1 26415 22504 33048 18400
23	2 52571 80000 97946 90880	3 46267 79033 60088 50400
24	6 41953 32502 48948 39320	8 94525 12503 46895 30200
25	15 40687 98005 97476 14368	21 82641 30508 46424 53688
26	34 96176 57013 55811 24912	50 36864 55019 53287 39280
27	75 10305 22473 56927 86848	110 06481 79487 12739 11760
28	152 88835 63606 90888 87512	227 99140 86080 47816 74360
29	295 23268 81447 82406 10368	448 12104 45054 73294 97880
30	541 25992 82654 34411 19008	836 49261 64102 16817 29376
31	942 83987 50430 14780 78272	1484 09980 33084 49191 97280
32	1561 57854 30399 93230 67138	2504 41841 80830 08011 45410
33	2460 66921 93357 46908 93672	4022 24776 23757 40139 60810
34	3691 00382 90036 20363 40508	6151 67304 83393 67272 34180
35	5272 86261 28623 14804 86440	8963 86644 18659 35168 26948
36	7176 95188 97292 61817 73210	12449 81450 25915 76622 59650
37	9310 64028 93785 01817 59840	16487 59217 91077 63635 33050
38	11515 79193 68628 83827 02960	20826 43222 62413 85644 62800
39	13582 72895 11716 06565 21440	25098 52088 80344 90392 24400
40	15280 57007 00680 57385 86620	28863 29902 12396 63951 08060
41	16398 66056 29998 66462 88080	31679 23063 30679 23848 74700
42	16789 10486 21189 10902 47320	33187 76542 51187 77365 35400

m	$\binom{86}{m}$	$\binom{87}{m}$
0	1	1
1	86	87
2	3655	3741
3	1 02340	1 05995
4	21 23555	22 25895
5	348 26302	369 49857
6	4701 55077	5049 81379
7	53732 00880	58433 55957
8	5 30603 58690	5 84335 59570
9	45 98564 41980	51 29168 00670
10	354 08946 03246	400 07510 45226
11	2446 43627 13336	2800 52573 16582
12	15290 22669 58350	17736 66296 71686
13	87036 67503 78300	1 02326 90173 36650
14	4 53834 09126 86850	5 40870 76630 65150
15	21 78403 63808 96880	26 32237 72935 83730
16	96 66666 14402 29905	118 45069 78211 26785
17	398 03919 41656 52550	494 70585 56058 82455
18	1525 81691 09683 34775	1923 85610 51339 87325
19	5460 81841 82024 61300	6986 63532 91707 96075
20	18293 74170 09782 45355	23754 56011 91807 06655
21	57494 61677 45030 56830	75788 35847 54813 02185
22	1 69870 45865 19408 49725	2 27365 07542 64439 06555
23	4 72683 01537 93136 68800	6 42553 47403 12545 18525
24	12 40792 91537 06983 80600	17 13475 93075 00120 49400
25	30 77166 43011 93319 83888	43 17959 34549 00303 64488
26	72 19505 85527 99711 92968	102 96672 28539 93031 76856
27	160 43346 34506 66026 51040	232 62852 20034 65738 44008
28	338 05622 65567 60555 86120	498 48969 00074 26582 37160
29	676 11245 31135 21111 72240	1014 16867 96702 81667 58360
30	1284 61366 09156 90112 27256	1960 72611 40292 11223 99496
31	2320 59241 97186 66009 26656	3605 20608 06343 56121 53912
32	3988 51822 13914 57203 42690	6309 11064 11101 23212 69346
33	6526 66618 04587 48151 06220	10515 18440 18502 05354 48910
34	10173 92081 07151 07411 94990	16700 58699 11738 55563 01210
35	15115 53949 02053 02440 61128	25289 46030 09204 09852 56118
36	21413 68094 44575 11790 86598	36529 22043 46628 14231 47726
37	28937 40668 16993 40257 92700	50351 08762 61568 52048 79298
38	37314 02440 53491 49279 95850	66251 43108 70484 89537 88550
39	45924 95311 42758 76036 87200	83238 97751 96250 25316 83050
40	53961 81990 92741 54343 32460	99886 77302 35500 30380 19660
41	60542 52965 43075 87799 82760	1 14504 34956 35817 42143 15220
42	64866 99605 81867 01214 10100	1 25409 52571 24942 89013 92860
43	66375 53085 02375 54730 70800	1 31242 52690 84242 55944 80900

Binomial Coefficients $n = 88(1)89$

m	$\binom{88}{m}$	$\binom{89}{m}$
0	1	1
1	88	89
2	3828	3916
3	1 09736	1 13564
4	23 31890	24 41626
5	391 75752	415 07642
6	5419 31236	5811 06988
7	63483 37336	68902 68572
8	6 42769 15527	7 06252 52863
9	57 13503 60240	63 56272 75767
10	451 36678 45896	508 50182 06136
11	3200 60083 61808	3651 96762 07704
12	20537 18869 88268	23737 78953 50076
13	1 20063 56470 08336	1 40600 75339 96604
14	6 43197 66804 01800	7 63261 23274 10136
15	31 73108 49566 48880	38 16306 16370 50680
16	144 77307 51147 10515	176 50416 00713 59395
17	613 15655 34270 09240	757 92962 85417 19755
18	2418 56196 07398 69780	3031 71851 41668 79020
19	8910 49143 43047 83400	11329 05339 50446 53180
20	30741 19544 83515 02730	39651 68688 26562 86130
21	99542 91859 46620 08840	1 30284 11404 30135 11570
22	3 03153 43390 19252 08740	4 02696 35249 65872 17580
23	8 69918 54945 76984 25080	11 73071 98335 96236 33820
24	23 56029 40478 12665 67925	32 25947 95423 89649 93005
25	60 31435 27624 00424 13888	83 87464 68102 13089 81813
26	146 14631 63088 93335 41344	206 46066 90712 93759 55232
27	335 59524 48574 58770 20864	481 74156 11663 52105 62208
28	731 11821 20108 92320 81168	1066 71345 68683 51091 02032
29	1512 65836 96777 08249 95520	2243 77658 16886 00570 76688
30	2974 89479 36994 92891 57856	4487 55316 33772 01141 53376
31	5565 93219 46635 67345 53408	8540 82698 83630 60237 11264
32	9914 31672 17444 79334 23258	15480 24891 64080 46679 76666
33	16824 29504 29603 28567 18256	26738 61176 47048 07901 41514
34	27215 77139 30240 60917 50120	44040 06643 59843 89484 68376
35	41990 04729 20942 65415 57328	69205 81868 51183 26333 07448
36	61818 68073 55832 24084 03844	1 03808 72802 76774 89499 61172
37	86880 30806 08196 66280 27024	1 48698 98879 64028 90364 30868
38	1 16602 51871 32053 41586 67848	2 03482 82677 40250 07866 94872
39	1 49490 40860 66735 14854 71600	2 66092 92731 98788 56441 39448
40	1 83125 75054 31750 55697 02710	3 32616 15914 98485 70551 74310
41	2 14391 12258 71317 72523 34880	3 97516 87313 03068 28220 37590
42	2 39913 87527 60760 31157 08080	4 54304 99786 32078 03680 42960
43	2 56652 05262 09185 44958 73760	4 96565 92789 69945 76115 81840
44	2 62485 05381 68485 11889 61800	5 19137 10643 77670 56848 35560

m	$\binom{90}{m}$	$\binom{91}{m}$
0	1	1
1	90	91
2	4005	4095
3	1 17480	1 21485
4	25 55190	26 72670
5	439 49268	465 04458
6	6226 14630	6665 63898
7	74713 75560	80939 90190
8	7 75155 21435	8 49868 96995
9	70 62525 28630	78 37680 50065
10	572 06454 81903	642 68980 10533
11	4160 46944 13840	4732 53398 95743
12	27389 75715 57780	31550 22659 71620
13	1 64338 54293 46680	1 91728 30009 04460
14	9 03861 98614 06740	10 68200 52907 53420
15	45 79567 39644 60816	54 83429 38258 67556
16	214 66722 17084 10075	260 46289 56728 70891
17	934 43378 86130 79150	1149 10101 03214 89225
18	3789 64814 27085 98775	4724 08193 13216 77925
19	14360 77190 92115 32200	18150 42005 19201 30975
20	50980 74027 77009 39310	65341 51218 69124 71510
21	1 69935 80092 56697 97700	2 20916 54120 33707 37010
22	5 32980 46653 96007 29150	7 02916 26746 52705 26850
23	15 75768 33585 62108 51400	21 08748 80239 58115 80550
24	43 99019 93759 85886 26825	59 74788 27345 47994 78225
25	116 13412 63526 02739 74818	160 12432 57285 88626 01643
26	290 33531 58815 06849 37045	406 46944 22341 09589 11863
27	688 20223 02376 45865 17440	978 53754 61191 52714 54485
28	1548 45501 80347 03196 64240	2236 65724 82723 49061 81680
29	3310 49003 85569 51661 78720	4858 94505 65916 54858 42960
30	6731 32974 50658 01712 30064	10041 81978 36227 53374 08784
31	13028 38015 17402 61378 64640	19759 70989 68060 63090 94704
32	24021 07590 47711 06916 87930	37049 45605 65113 68295 52570
33	42218 86068 11128 54581 18180	66239 93658 58839 61498 06110
34	70778 67820 06891 97386 09890	1 12997 53888 18020 51967 28070
35	1 13245 88512 11027 15817 75824	1 84024 56332 17919 13203 85714
36	1 73014 54671 27958 15832 68620	2 86260 43183 38985 31650 44444
37	2 52507 71682 40803 79863 92040	4 25522 26353 68761 95696 60660
38	3 52181 81557 04278 98231 25740	6 04689 53239 45082 78095 17780
39	4 69575 75409 39038 64308 34320	8 21757 56966 43317 62539 60060
40	5 98709 08646 97274 26993 13758	10 68284 84056 36312 91301 48078
41	7 30133 03228 01553 98772 11900	13 28842 11874 98828 25765 25658
42	8 51821 87099 35146 31900 80550	15 81954 90327 36700 30672 92450
43	9 50870 92576 02023 79796 24800	18 02692 79675 37170 11697 05350
44	10 15703 03433 47616 32964 17400	19 66573 96009 49640 12760 42200
45	10 38274 21287 55341 13696 71120	20 53977 24721 02957 46660 88520

Binomial Coefficients $n = 92(1)93$

m	$\binom{92}{m}$	$\binom{93}{m}$
0	1	1
1	92	93
2	4186	4278
3	1 25580	1 29766
4	27 94155	29 19735
5	491 77128	519 71283
6	7130 68356	7622 45484
7	87605 54088	94736 22444
8	9 30808 87185	10 18414 41273
9	86 87549 47060	96 18358 34245
10	721 06660 60598	807 94210 07658
11	5375 22379 06276	6096 29039 66874
12	36282 76058 67363	41657 98437 73639
13	2 23278 52668 76080	2 59561 28727 43443
14	12 59928 82916 57880	14 83207 35585 33960
15	65 51629 91166 20976	78 11558 74082 78856
16	315 29718 94987 38447	380 81348 86153 59423
17	1409 56390 59943 60116	1724 86109 54930 98563
18	5873 18294 16431 67150	7282 74684 76375 27266
19	22874 50198 32418 08900	28747 68492 48849 76050
20	83491 93223 88326 02485	1 06366 43422 20744 11385
21	2 86258 05339 02832 08520	3 69749 98562 91158 11005
22	9 23832 80866 86412 63860	12 10090 86205 89244 72380
23	28 11665 06986 10821 07400	37 35497 87852 97233 71260
24	80 83537 07585 06110 58775	108 95202 14571 16931 66175
25	219 87220 84631 36620 79868	300 70757 92216 42731 38643
26	566 59376 79626 98215 13506	786 46597 64258 34835 93374
27	1385 00698 83532 62303 66348	1951 60075 63159 60518 79854
28	3215 19479 43915 01776 36165	4600 20178 27447 64080 02513
29	7095 60230 48640 03920 24640	10310 79709 92555 05696 60805
30	14900 76484 02144 08232 51744	21996 36714 50784 12152 76384
31	29801 52968 04288 16465 03488	44702 29452 06432 24697 55232
32	56809 16595 33174 31386 47274	86610 69563 37462 47851 50762
33	1 03289 39264 23953 29793 58680	1 60098 55859 57127 61180 05954
34	1 79237 47546 76860 13465 34180	2 82526 86811 00813 43258 92860
35	2 97022 10220 35939 65171 13784	4 76259 57767 12799 78636 47964
36	4 70284 99515 56904 44854 30158	7 67307 09735 92844 10025 43942
37	7 11782 69537 07747 27347 05104	11 82067 69052 64651 72201 35262
38	10 30211 79593 13844 73791 78440	17 41994 49130 21592 01138 83544
39	14 26447 10205 88400 40634 77840	24 56658 89799 02245 14426 56280
40	18 90042 41022 79630 53841 08138	33 16489 51228 68030 94475 85978
41	23 97126 95931 35141 17066 73736	42 87169 36954 14771 70907 81874
42	29 10797 02202 35528 56438 18108	53 07923 98133 70669 73504 91844
43	33 84647 70002 73870 42369 97800	62 95444 72205 09398 98808 15908
44	37 69266 75684 86810 24457 47550	71 53914 45687 60680 66827 45350
45	40 20551 20730 52597 59421 30720	77 89817 96415 39407 83878 78270
46	41 07954 49442 05914 93321 77040	81 28505 70172 58512 52743 07760

m	$\binom{94}{m}$	$\binom{95}{m}$
0	1	1
1	94	95
2	4371	4465
3	1 34044	1 38415
4	30 49501	31 83545
5	548 91018	579 40519
6	8142 16767	8691 07785
7	1 02358 67928	1 10500 84695
8	11 13150 63717	12 15509 31645
9	106 36772 75518	117 49923 39235
10	904 12568 41903	1010 49341 17421
11	6904 23249 74532	7808 35818 16435
12	47754 27477 40513	54658 50727 15045
13	3 01219 27165 17082	3 48973 54642 57595
14	17 42768 64312 77403	20 43987 91477 94485
15	92 94766 09668 12816	110 37534 73980 90219
16	458 92907 60236 38279	551 87673 69904 51095
17	2105 67458 41084 57986	2564 60366 01320 96265
18	9007 60794 31306 25829	11113 28252 72390 83815
19	36030 43177 25225 03316	45038 03971 56531 29145
20	1 35114 11914 69593 87435	1 71144 55091 94818 90751
21	4 76116 41985 11902 22390	6 11230 53899 81496 09825
22	15 79840 84768 80402 83385	20 55957 26753 92305 05775
23	49 45588 74058 86478 43640	65 25429 58827 66881 27025
24	146 30700 02424 14165 37435	195 76288 76483 00643 81075
25	409 65960 06787 59663 04818	555 96660 09211 73828 42253
26	1087 17355 56474 77567 32017	1496 83315 63262 37230 36835
27	2738 06673 27417 95354 73228	3825 24028 83892 72922 05245
28	6551 80253 90607 24598 82367	9289 86927 18025 19953 55595
29	14910 99888 20002 69776 63318	21462 80142 10609 94375 45685
30	32307 16424 43339 17849 37189	47218 16312 63341 87626 00507
31	66698 66166 57216 36850 31616	99005 82591 00555 54699 68805
32	1 31312 99015 43894 72549 05994	1 98011 65182 01111 09399 37610
33	2 46709 25422 94590 09031 56716	3 78022 24438 38484 81580 62710
34	4 42625 42670 57941 04438 98814	6 89334 68093 52531 13470 55530
35	7 58786 44578 13613 21895 40824	12 01411 87248 71554 26334 39638
36	12 43566 67503 05643 88661 91906	20 02353 12081 19257 10557 32730
37	19 49374 78788 57495 82226 79204	31 92941 46291 63139 70888 71110
38	29 24062 18182 86243 73340 18806	48 73436 96971 43739 55566 98010
39	41 98653 38929 23837 15565 39824	71 22715 57112 10080 88905 58630
40	57 73148 41027 70276 08902 42258	99 71801 79956 94113 24467 82082
41	76 03658 88182 82802 65383 67852	133 76807 29210 53078 74286 10110
42	95 95093 35087 85441 44412 73718	171 98752 23270 68244 09796 41570
43	116 03368 70338 80068 72313 07752	211 98462 05426 65510 16725 81470
44	134 49359 17892 70079 65635 61258	250 52727 88231 50148 37948 69010
45	149 43732 42103 00088 50706 23620	283 93091 59995 70168 16341 84878
46	159 18323 66587 97920 36621 86030	308 62056 08690 98008 87328 09650
47	162 57011 40345 17025 05486 15520	321 75335 06933 14945 42108 01550

Binomial Coefficients $n=96(1)97$

m	$\binom{96}{m}$	$\binom{97}{m}$
0	1	1
1	96	97
2	4560	4656
3	1 42880	1 47440
4	33 21960	34 64840
5	611 24064	644 46024
6	9270 48304	9881 72368
7	1 19191 92480	1 28462 40784
8	13 26010 16340	14 45202 08820
9	129 65432 70880	142 91442 87220
10	1127 99264 56656	1257 64697 27536
11	8818 85159 33856	9946 84423 90512
12	62466 86545 31480	71285 71704 65336
13	4 03632 05369 72640	4 66098 91915 04120
14	23 92961 46120 52080	27 96593 51490 24720
15	130 81522 65458 84704	154 74484 11579 36784
16	662 25208 43885 41314	793 06731 09344 26018
17	3116 48039 71225 47360	3778 73248 15110 88674
18	13677 88618 73711 80080	16794 36658 44937 27440
19	56151 32224 28922 12960	69829 20843 02633 93040
20	2 16182 59063 51350 19896	2 72333 91287 80272 32856
21	7 82375 08991 76315 00576	9 98557 68055 27665 20472
22	26 67187 80653 73801 15600	34 49562 89645 50116 16176
23	85 81386 85581 59186 32800	112 48574 66235 32987 48400
24	261 01718 35310 67525 08100	346 83105 20892 26711 40900
25	751 72948 85694 74472 23328	1012 74667 21005 41997 31428
26	2052 79975 72474 11058 79088	2804 52924 58168 85531 02416
27	5322 07344 47155 10152 42080	7374 87320 19629 21211 21168
28	13115 10956 01917 92875 60840	18437 18300 49073 03028 02920
29	30752 67069 28635 14329 01280	43867 78025 30553 07204 62120
30	68680 96454 73951 82001 46192	99433 63524 02586 96330 47472
31	1 46223 98903 63897 42325 69312	2 14904 95358 37849 24327 15504
32	2 97017 47773 01666 64099 06415	4 43241 46676 65564 06424 75727
33	5 76033 89620 39595 90980 00320	8 73051 37393 41262 55079 06735
34	10 67356 92531 91015 95051 18240	16 43390 82152 30611 86031 18560
35	18 90746 55342 24085 39804 95168	29 58103 47874 15101 34856 13408
36	32 03764 99329 90811 36891 72368	50 94511 54672 14896 76696 67536
37	51 95294 58372 82396 81446 03840	83 99059 57702 73208 18337 76208
38	80 66378 43263 06879 26455 69120	132 61673 01635 89276 07901 72960
39	119 96152 54083 53820 44472 56640	200 62530 97346 60699 70928 25760
40	170 94517 37069 04194 13373 40712	290 90669 91152 58014 57845 97352
41	233 48609 09167 47191 98753 92192	404 43126 46236 51386 12127 32904
42	305 75559 52481 21322 84082 51680	539 24168 61648 68514 82836 43872
43	383 97214 28697 33754 26522 23040	689 72773 81178 55077 10604 74720
44	462 51189 93658 15658 54674 50480	846 48404 22355 49412 81196 73520
45	534 45819 48227 20316 54290 53888	996 97009 41885 35975 08965 04368
46	592 55147 68686 68177 03669 94528	1127 00967 16913 88493 57960 48416
47	630 37391 15624 12954 29436 11200	1222 92538 84310 81131 33106 05728
48	643 50670 13866 29890 84216 03100	1273 88061 29490 42845 13652 14300

m	$\binom{98}{m}$	$\binom{99}{m}$
0	1	1
1	98	99
2	4753	4851
3	1 52096	1 56849
4	36 12280	37 64376
5	679 10864	715 23144
6	10526 18392	11205 29256
7	1 38344 13152	1 48870 31544
8	15 73664 49604	17 12008 62756
9	157 36644 96040	173 10309 45644
10	1400 56140 14756	1557 92785 10796
11	11204 49121 18048	12605 05261 32804
12	81232 56128 55848	92437 05249 73896
13	5 37384 63619 69456	6 18617 19748 25304
14	32 62692 43405 28840	38 00077 07024 98296
15	182 71077 63069 61504	215 33770 06474 90344
16	947 81215 20923 62802	1130 52292 83993 24306
17	4571 79979 24455 14692	5519 61194 45378 77494
18	20573 09906 60048 16114	25144 89885 84503 30806
19	86623 57501 47571 20480	1 07196 67408 07619 36594
20	3 42163 12130 82906 25896	4 28786 69632 30477 46376
21	12 70891 59343 07937 53328	16 13054 71473 90843 79224
22	44 48120 57700 77781 36648	57 19012 17043 85718 89976
23	146 98137 55880 83103 64576	191 46258 13581 60885 01224
24	459 31679 87127 59698 89300	606 29817 43008 42802 53876
25	1359 57772 41897 68708 72328	1818 89452 29025 28407 61628
26	3817 27591 79174 27528 33844	5176 85364 21071 96237 06172
27	10179 40244 77798 06742 23584	13996 67836 56972 34270 57428
28	25812 05620 68702 24239 24088	35991 45865 46500 30981 47672
29	62304 96325 79626 10232 65040	88117 01946 48328 34471 89128
30	1 43301 41549 33140 03535 09592	2 05606 37875 12766 13767 74632
31	3 14338 58882 40436 20657 62976	4 57640 00431 73576 24192 72568
32	6 58146 42035 03413 30751 91231	9 72485 00917 43849 51409 54207
33	13 16292 84070 06826 61503 82462	19 74439 26105 10239 92255 73693
34	25 16442 19545 71874 41110 25295	38 32735 03615 78701 02614 07757
35	46 01494 30026 45713 20887 31968	71 17936 49572 17587 61997 57263
36	80 52615 02546 29998 11552 80944	126 54109 32572 75711 32440 12912
37	134 93571 12374 88104 95034 43744	215 46186 14921 18103 06587 24688
38	216 60732 59338 62484 26239 49168	351 54303 71713 50589 21273 92912
39	333 24203 98982 49975 78829 98720	549 84936 58321 12460 05069 47888
40	491 53200 88499 18714 28774 23112	824 77404 87481 68690 07604 21832
41	695 33796 37389 09400 69973 30256	1186 86997 25888 28114 98747 53368
42	943 67295 07885 19900 94963 76776	1639 01091 45274 29301 64937 07032
43	1228 96942 42827 23591 93441 18592	2172 64237 50712 43492 88404 95368
44	1536 21178 03534 04489 91801 48240	2765 18120 46361 28081 85242 66832
45	1843 45413 64240 85387 90161 77888	3379 66591 67774 89877 81963 26128
46	2123 97976 58799 24468 66925 52784	3967 43390 23040 09856 57087 30672
47	2349 93506 01224 69624 91066 54144	4473 91482 60023 94093 57992 06928
48	2496 80600 13801 23976 46758 20028	4846 74106 15025 93601 37824 74172
49	2547 76122 58980 85690 27304 28600	5044 56722 72782 09666 74062 48628

Binomial Coefficients $n = 100$

m	$\binom{100}{m}$
0	1
1	100
2	4950
3	1 61700
4	39 21225
5	752 87520
6	11920 52400
7	1 60075 60800
8	18 60878 94300
9	190 22318 08400
10	1731 03094 56440
11	14162 98046 43600
12	1 05042 10511 06700
13	7 11054 24997 99200
14	44 18694 26773 23600
15	253 33847 13499 88640
16	1345 86062 90468 14650
17	6650 13487 29372 01800
18	30664 51080 29882 08300
19	1 32341 57293 92122 67400
20	5 35983 37040 38096 82970
21	20 41841 41106 21321 25600
22	73 32066 88517 76562 69200
23	248 65270 30625 46603 91200
24	797 76075 56590 03687 55100
25	2425 19269 72033 71210 15504
26	6995 74816 50097 24644 67800
27	19173 53200 78044 30507 63600
28	49988 13702 03472 65252 05100
29	1 24108 47811 94828 65453 36800
30	2 93723 39821 61094 48239 63760
31	6 63246 38306 86342 37960 47200
32	14 30125 01349 17425 75602 26775
33	29 46924 27022 54089 43665 27900
34	58 07174 29720 88940 94869 81450
35	109 50671 53187 96288 64611 65020
36	197 72045 82144 93298 94437 70175
37	342 00295 47493 93814 39027 37600
38	567 00489 86634 68692 27861 17600
39	901 39240 30034 63049 26343 40800
40	1374 62341 45802 81150 12673 69720
41	2011 64402 13369 96805 06351 75200
42	2825 88088 71162 57416 63684 60400
43	3811 65328 95986 72794 53342 02400
44	4937 82357 97073 71574 73647 62200
45	6144 84712 14136 17959 67205 92960
46	7347 09981 90814 99734 39050 56800
47	8441 34872 83064 03950 15079 37600
48	9320 65588 75049 87694 95816 81100
49	9891 30828 87808 03268 11887 22800
50	10089 13445 45564 19333 48124 97256

Table of Stirling I Numbers $\mathbf{I}_n^{(m)}$ $n=1(1)40$

Stirling I Numbers $n=1(1)18$

m	$\mathbf{I}_1^{(m)}$	$\mathbf{I}_2^{(m)}$	$\mathbf{I}_3^{(m)}$	$\mathbf{I}_4^{(m)}$	$\mathbf{I}_5^{(m)}$	$\mathbf{I}_6^{(m)}$	$\mathbf{I}_7^{(m)}$	$\mathbf{I}_8^{(m)}$	$\mathbf{I}_9^{(m)}$	$\mathbf{I}_{10}^{(m)}$	$\mathbf{I}_{11}^{(m)}$
1	1	1	2	6	24	120	720	5040	40320	3 62880	36 28800
2		1	3	11	50	274	1764	13068	1 09584	10 26576	106 28640
3			1	6	35	225	1624	13132	1 18124	11 72700	127 53576
4				1	10	85	735	6769	67284	7 23680	84 09500
5					1	15	175	1960	22449	2 69325	34 16930
6						1	21	322	4536	63273	9 02055
7							1	28	546	9450	1 57773
8								1	36	870	18150
9									1	45	1320
10										1	55
11											1

m	$\mathbf{I}_{12}^{(m)}$	$\mathbf{I}_{13}^{(m)}$	$\mathbf{I}_{14}^{(m)}$	$\mathbf{I}_{15}^{(m)}$
1	399 16800	4790 01600	62270 20800	8 71782 91200
2	1205 43840	14864 42880	1 98027 59040	28 34656 47360
3	1509 17976	19315 59552	2 65967 17056	39 21567 97824
4	1052 58076	14140 14888	2 03137 53096	31 09892 60400
5	459 95730	6572 06836	99577 03756	15 97216 05680
6	133 39535	2060 70150	33361 18786	5 66633 66760
7	26 37558	449 90231	7909 43153	1 44093 22928
8	3 57423	69 26634	1350 36473	26814 53775
9	32670	7 49463	166 69653	3684 11615
10	1925	55770	14 74473	373 12275
11	66	2717	91091	27 49747
12	1	78	3731	1 43325
13		1	91	5005
14			1	105
15				1

m	$\mathbf{I}_{16}^{(m)}$	$\mathbf{I}_{17}^{(m)}$	$\mathbf{I}_{18}^{(m)}$
1	130 76743 68000	2092 27898 88000	35568 74280 96000
2	433 91630 01600	7073 42823 93600	1 22340 55905 79200
3	616 58176 14720	10299 22448 37120	1 82160 24446 24640
4	505 69957 03824	8707 77488 75904	1 58331 39757 27488
5	270 68133 45600	4836 60092 33424	90929 99058 44112
6	100 96721 07080	1886 15670 58880	36901 26492 34384
7	27 28032 10680	537 45234 77960	11022 84661 84200
8	5 46311 29553	114 69012 83528	2487 18452 97936
9	82076 28000	18 59531 77553	430 81053 01929
10	9280 95740	2 30571 59840	57 79248 94833
11	785 58480	21850 31420	6 02026 93980
12	48 99622	1569 52432	48532 22764
13	2 18400	83 94022	2996 50806
14	6580	3 23680	138 96582
15	120	8500	4 68180
16	1	136	10812
17		1	153
18			1

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Stirling I Numbers $n = 19(1)23$

m	$\mathbf{I}_{19}^{(m)}$	$\mathbf{I}_{20}^{(m)}$	$\mathbf{I}_{21}^{(m)}$
1	6 40237 37057 28000	121 64510 04088 32000	2432 90200 81766 40000
2	22 37698 80585 21600	431 56514 68176 38400	8752 94803 67616 00000
3	34 01224 95938 22720	668 60973 03411 53280	13803 75975 36407 04000
4	30 32125 40077 19424	610 11607 57404 91776	12870 93124 51509 88800
5	17 95071 22809 21504	371 38478 73452 28000	8037 81182 26450 51776
6	7 55152 75920 63024	161 42973 65301 18960	3599 97951 79476 07200
7	2 35312 50405 49984	52 26090 33625 12720	1206 64780 37803 73360
8	55792 16815 47048	12 95363 69899 43896	311 33364 31613 90640
9	10241 77407 32658	2 50385 87554 67550	63 03081 20992 94896
10	1471 07534 08923	38192 20555 02195	10 14229 98655 11450
11	166 15733 86473	4628 06477 51910	1 30753 50105 40395
12	14 75607 03732	446 52267 57381	13558 51828 99530
13	1 02469 37272	34 22525 11900	1131 02769 95381
14	5497 89282	2 06929 33630	75 61111 84500
15	223 23822	9739 41900	4 01717 71630
16	6 62796	349 16946	16722 80820
17	13566	9 20550	533 27946
18	171	16815	12 56850
19	1	190	20615
20		1	210
21			1

m	$\mathbf{I}_{22}^{(m)}$	$\mathbf{I}_{23}^{(m)}$
1	51090 94217 17094 40000	11 24000 72777 76076 80000
2	1 86244 81078 01702 40000	41 48476 77933 54547 20000
3	2 98631 90286 32163 84000	67 56146 67377 09306 88000
4	2 84093 31590 18114 68800	65 48684 85270 30686 97600
5	1 81664 97952 06970 76096	42 80722 86535 71471 42912
6	83637 38169 95448 02976	20 21687 37691 06827 41568
7	28939 58339 73354 47760	7 20308 21644 09246 53696
8	7744 65431 01695 76800	1 99321 97822 10661 37360
9	1634 98069 72465 83456	43714 22964 95944 12832
10	276 01910 92750 35346	7707 40110 12973 61068
11	37 60053 50868 59745	1103 23088 11859 49736
12	4 15482 38514 30525	129 00665 98183 31295
13	37310 09998 02531	12 36304 58470 86207
14	2718 86118 69881	97125 04609 39913
15	159 97183 88730	6238 24164 21941
16	7 52896 68850	325 60911 03430
17	27921 67686	13 67173 57942
18	797 21796	45460 47198
19	16 89765	1168 96626
20	25025	22 40315
21	231	30107
22	1	253
23		1

m	$\mathbf{I}_{24}^{(m)}$	$\mathbf{I}_{25}^{(m)}$
1	258 52016 73888 49766 40000	6204 48401 73323 94393 60000
2	965 38966 65249 30662 40000	23427 87216 39871 85664 00000
3	1595 39850 27606 68605 44000	39254 95373 27809 77192 96000
4	1573 75898 28594 15107 32800	39365 61409 13866 31181 31200
5	1050 05310 75591 74529 84576	26775 03356 42796 03823 62624
6	507 79532 53430 28501 98976	13237 14091 57918 58577 60000
7	185 88776 35505 19497 76576	4969 10165 05554 96448 36800
8	53 04713 71552 54458 12976	1459 01905 52766 26492 88000
9	12 04749 26016 17376 32496	342 18695 95940 71489 92880
10	2 20984 45497 94337 17396	65 08376 17966 81468 50000
11	33081 71136 85742 04996	10 14945 52782 52146 37300
12	4070 38405 70075 69521	1 30770 92873 67558 73500
13	413 35671 43013 14056	13990 94520 02391 06865
14	34 70180 64487 04206	1246 20006 90702 15000
15	2 40604 60386 44556	92 44691 13761 73550
16	13727 25118 00831	5 70058 63218 64500
17	640 05903 36096	29088 66798 67135
18	24 12764 43496	1219 12249 80000
19	72346 69596	41 49085 13800
20	1684 23871	1 12768 42500
21	29 32776	2388 10495
22	35926	37 95000
23	276	42550
24	1	300
25		1

m	$\mathbf{I}_{26}^{(m)}$	$\mathbf{I}_{27}^{(m)}$
1	1 55112 10043 33098 59840 00000	40 32914 61126 60563 55840 00000
2	5 91901 28811 70120 35993 60000	155 44545 59147 56227 95673 60000
3	10 04801 71548 35116 15488 00000	267 16745 89068 83140 38681 60000
4	10 23395 30601 74467 56725 76000	276 13079 67193 71272 90357 76000
5	7 08741 45319 83767 26771 96800	194 50673 08917 52416 52796 92800
6	3 57703 55645 90760 68263 62624	100 09033 92113 43545 01626 25024
7	1 37464 68217 96792 69786 80000	39 31785 29313 07370 82720 42624
8	41444 57803 24711 58770 36800	12 15023 71102 39293 97816 36800
9	10013 69304 51284 13741 10000	3 01800 59720 58099 16038 96800
10	1969 28100 45111 08202 42880	61214 99916 24172 27004 24880
11	318 82014 37529 85127 82500	10258 60474 20887 21525 87880
12	42 84218 74624 41114 74800	1432 71701 77764 54111 27300
13	4 80544 55874 27335 45125	167 78377 27355 51836 48050
14	45145 94692 69944 81865	16 54339 17884 45900 73615
15	3557 37285 34745 53750	1 37637 64111 73328 79365
16	234 96156 94227 86050	9666 37365 84669 91050
17	12 97275 33185 42875	572 25315 57049 00800
18	59566 73043 67135	28 46010 32320 88385
19	2256 39378 25000	1 18232 96878 17135
20	69 68295 76300	4068 15068 08800
21	1 72471 04875	114 52543 03050
22	3336 85495	2 59229 27745
23	48 58750	4600 12995
24	50050	61 60050
25	325	58500
26	1	351
27		1

Stirling I Numbers $n = 28$

m	$\mathbf{I}_{28}^{(m)}$
1	1088 88694 50418 35216 07680 00000
2	4237 35645 58110 78718 39027 20000
3	7368 96684 64006 01018 40076 80000
4	7722 69897 03299 07508 78341 12000
5	5527 81253 07966 86519 15874 81600
6	2896 94588 95980 28131 96705 68448
7	1161 67236 83566 42557 35077 75872
8	367 37425 49077 68308 23762 36224
9	93 63639 83558 07971 30868 50400
10	19 54605 57459 10750 45153 68560
11	3 38197 32719 88127 08202 97640
12	48941 96422 20529 82530 24980
13	5962 87888 16363 53696 24650
14	614 45535 10235 91156 35655
15	53 70555 48901 25778 16470
16	3 98629 72989 59416 37715
17	25117 20886 24993 12650
18	1340 67594 29712 87195
19	60 38300 48031 51030
20	2 28073 03716 54735
21	7160 33729 91150
22	184 51733 52165
23	3 83432 78610
24	6263 34345
25	77 39550
26	67977
27	378
28	1

m	$I_{29}^{(m)}$
1	30488 83446 11713 86050 15040 00000
2	1 19734 86770 77520 39331 00441 60000
3	2 10568 42815 50279 07233 61177 60000
4	2 23604 53801 56380 11264 33628 16000
5	1 62501 44983 26371 30045 22835 96800
6	86642 29743 95414 74214 23633 98144
7	35423 77220 35840 19737 78882 92864
8	11448 15150 57741 55188 00423 90144
9	2989 19340 88703 91504 88080 47424
10	640 92595 92413 08983 95171 70080
11	114 24130 73615 78308 74837 02480
12	17 08572 32541 62962 19049 97080
13	2 15902 57290 78708 86025 15180
14	23167 62871 02969 06074 22990
15	2118 21088 79471 12944 96815
16	165 32187 92609 89436 72490
17	11 01911 57804 59223 91915
18	62656 13526 56953 54110
19	3031 40007 74595 16035
20	124 24345 52094 83610
21	4 28562 48154 06935
22	12326 82268 51770
23	291 87851 53245
24	5 58806 40270
25	8430 41745
26	96 42906
27	78561
28	406
29	1

Stirling I Numbers $n = 30$

m	$I_{30}^{(m)}$
1	8 84176 19937 39701 95454 36160 00000
2	35 02799 99798 59805 26649 27846 40000
3	62 26219 28420 35613 49105 74592 00000
4	66 95100 03060 85302 33899 36394 24000
5	49 36146 58316 21147 82575 95871 23200
6	26 75128 07557 93398 82258 08221 42976
7	11 13931 69134 34780 46610 11238 91200
8	3 67420 16587 10345 20189 91176 07040
9	98134 76036 30155 08829 54757 65440
10	21576 04622 68683 52039 48059 79744
11	3953 92387 27270 79937 65445 42000
12	609 72728 17323 04212 27286 17800
13	79 69746 93974 45519 13779 37300
14	8 87763 80550 64811 62177 81890
15	84595 74446 07631 81478 30625
16	6912 54538 65158 06609 99025
17	484 87623 68943 06930 38025
18	29 18939 50075 10876 61105
19	1 50566 73751 20213 19125
20	6634 46027 85345 40725
21	248 52657 48562 84725
22	7 86040 33941 08265
23	20791 29962 95875
24	453 93237 21075
25	8 03288 50875
26	11226 86019
27	119 21175
28	90335
29	435
30	1

m	$\mathbf{I}_{31}^{(m)}$
1	265 25285 98121 91058 63630 84800 00000
2	1059 68176 13895 33859 94932 71552 00000
3	1902 89378 52409 28209 99821 65606 40000
4	2070 79220 20245 94683 66086 66419 20000
5	1547 79497 52547 19737 11178 12531 20000
6	851 89988 85054 23112 50318 42514 12480
7	360 93078 81588 36812 80561 45388 78976
8	121 36536 66747 45136 52307 46521 02400
9	33 11462 97676 14997 85076 33905 70240
10	7 45416 14716 90660 70013 96551 57760
11	1 40193 76240 86807 50169 11422 39744
12	22245 74232 46962 06305 84030 76000
13	3000 65136 36556 69786 40667 36800
14	346 02661 10493 89867 79113 94000
15	34 25636 13932 93766 06527 00640
16	2 91972 10605 62373 79778 01375
17	21458 83249 33450 14521 39775
18	1360 55808 71196 33228 71175
19	74 35941 62611 17272 34855
20	3 49600 54586 80575 40875
21	14090 25752 42230 82475
22	484 33867 66795 32675
23	14 09779 32829 84515
24	34409 27079 28125
25	694 91892 47325
26	11 40094 31445
27	14803 21269
28	146 31225
29	1 03385
30	465
31	1

Stirling I Numbers $n=32$

m	$\mathbf{I}_{32}^{(m)}$
1	8222 83865 41779 22817 72556 28800 00000
2	33115 38746 28877 40717 06545 02912 00000
3	60049 38910 38583 08369 89404 05350 40000
4	66097 45204 80033 63403 48508 24601 60000
5	50052 43643 49209 06534 12608 54886 40000
6	27956 69151 89228 36224 71049 30469 06880
7	12040 75432 14293 64309 47723 49566 60736
8	4123 25715 50759 36045 02092 87540 53376
9	1147 91888 94708 10069 89673 97597 79840
10	264 19363 53900 25479 55509 27004 60800
11	50 91422 78183 81693 25256 50645 89824
12	8 29811 77447 42631 45650 16375 95744
13	1 15265 93459 80219 69684 44719 16800
14	13727 47630 61867 55687 93199 50800
15	1407 97381 42414 96615 81451 13840
16	124 76771 42707 27353 79645 43265
17	9 57195 91334 99328 29941 34400
18	63636 13319 40536 44611 46200
19	3665 69999 12142 68671 51680
20	182 73558 54802 15110 01980
21	7 86398 52911 89730 97600
22	29104 75650 12885 95400
23	921 37026 84520 52640
24	24 76466 72287 56390
25	55951 75745 95200
26	1048 34816 22120
27	15 98993 90784
28	19338 89244
29	178 36160
30	1 17800
31	496
32	1

m	$I_{33}^{(m)}$
1	2 63130 83693 36935 30167 21801 21600 00000
2	10 67915 23746 65856 25763 81997 21984 00000
3	19 54695 83878 63536 08553 67474 74124 80000
4	21 75167 85463 99659 37281 41667 92601 60000
5	16 67775 41796 54723 72495 51981 80966 40000
6	9 44666 56504 04516 65724 86186 29896 60160
7	4 13260 82980 46624 94127 98201 16600 50432
8	1 43984 98328 38593 17750 14695 50863 68768
9	40856 66161 81418 58281 71660 10670 08256
10	9602 11522 19516 25415 65970 61745 25440
11	1893 44892 55782 39663 63717 47673 35168
12	316 45399 56501 45899 86061 74676 53632
13	45 18321 68161 09661 75552 47389 33344
14	5 54545 17639 59981 51698 27103 42400
15	58782 63836 19146 47393 99635 93680
16	5400 54067 09047 71937 30104 98320
17	431 07040 65427 05859 37768 44065
18	29 93552 17555 96494 57508 12800
19	1 80938 53291 29102 42099 99960
20	9513 23872 65811 52192 15040
21	434 38311 47982 86501 25180
22	17 17750 73716 02081 50400
23	58588 60509 17542 79880
24	1713 83961 97722 57120
25	42 66922 96158 02790
26	89498 89865 03040
27	1560 02621 27208
28	22 17838 46592
29	25046 46364
30	216 05760
31	1 33672
32	528
33	1

Stirling I Numbers $n = 34$

m	$I_{34}^{(m)}$
1	86 83317 61881 18864 95518 19440 12800 00000
2	355 04333 67333 10191 80373 27709 47072 00000
3	655 72877 91741 62547 08035 08663 68102 40000
4	737 35235 04190 52295 38840 42516 29977 60000
5	572 11756 64750 05542 29633 57067 64492 80000
6	328 41772 06430 03773 41415 96129 67554 25280
7	145 82273 94859 43139 71948 26824 77713 24416
8	51 64765 27817 20199 79882 83152 95102 19776
9	14 92254 81668 25406 41046 79479 02976 41216
10	3 57726 46394 25454 96998 48690 48263 47776
11	72085 92976 60335 34315 68647 34965 85984
12	12336 43078 20330 54359 03755 11999 05024
13	1807 50015 05817 64737 79293 38524 53984
14	228 18312 50267 89051 81595 41802 32544
15	24 94372 24233 91815 15700 15089 33840
16	2 37000 48050 17721 21324 93100 38240
17	19625 86408 68140 65296 76463 52465
18	1418 94262 44773 90180 35536 66465
19	89 64523 76168 56874 46808 11480
20	4 94875 41089 00882 64440 96280
21	23847 88151 49246 06733 45980
22	1001 24085 80611 55190 88380
23	36 51174 70518 80993 86440
24	1 15145 31254 42387 64840
25	3121 92419 70937 49190
26	72 20386 61704 03110
27	1 40979 76367 00904
28	2291 91290 64744
29	30 44371 76604
30	32176 36444
31	260 16936
32	1 51096
33	561
34	1

m	$\mathbf{I}_{35}^{(m)}$
1	2952 32799 03960 41408 47618 60964 35200 00000
2	12158 30662 51206 65386 28209 61562 13248 00000
3	22649 82182 86548 36792 53566 22274 62553 60000
4	25725 70869 34219 40590 28609 54217 87340 80000
5	20189 34961 05692 40733 46381 82816 22732 80000
6	11738 32006 83371 33838 37776 25476 61337 39520
7	5286 39086 31650 70523 87657 08172 09804 55424
8	1901 84293 40644 29932 87964 54025 11187 96800
9	559 01429 04537 84017 75473 85439 96300 21120
10	136 54954 59072 90875 38995 34955 43934 65600
11	28 08648 07598 76856 63731 82700 37102 71232
12	4 91524 57635 51573 82522 96321 42933 56800
13	73791 43590 18130 55443 99730 21833 40480
14	9565 72640 14925 92499 53537 59803 60480
15	1076 26968 74221 10767 15400 54839 83104
16	105 52388 57939 94336 40747 80502 34000
17	9 04279 85945 34503 41414 92860 22050
18	67869 91331 90453 31428 84710 12275
19	4466 88070 34505 23912 27012 56785
20	257 90287 73194 86884 37800 85000
21	13 05703 38239 75248 93378 59600
22	57890 07068 90038 83223 50900
23	2242 64025 78251 08982 27340
24	75 66115 33169 22173 91000
25	2 21290 73524 54262 37300
26	5576 85564 68874 54930
27	120 13698 58182 33846
28	2 18904 80249 02200
29	3326 99930 69280
30	41 38368 15700
31	41022 12268
32	311 54200
33	1 70170
34	595
35	1

Stirling **I** Numbers $n = 36$

m	$\mathbf{I}^{(m)}_{36}$
1	1 03331 47966 38614 49296 66651 33752 32000 00000
2	4 28493 05986 96193 29928 34955 15638 98880 00000
3	8 04902 07062 80399 53125 03027 41174 02624 00000
4	9 23049 62609 84227 57452 54900 19900 19481 60000
5	7 32352 94506 33453 66261 51973 52785 82988 80000
6	4 31030 55200 23689 25076 68550 74497 69541 63200
7	1 96762 00027 91146 02174 05774 11500 04496 79360
8	71850 89355 54201 18174 66415 99051 01383 43424
9	21467 34309 99468 70554 29549 44423 81695 36000
10	5338 24839 72089 64656 40311 08880 34013 17120
11	1119 57637 25029 80857 69609 29468 42529 58720
12	200 12008 24841 81940 52035 53950 39777 59232
13	30 74224 83291 86143 23062 86879 07102 73600
14	4 08591 85995 40537 92927 73546 14959 57280
15	47235 16546 12664 69349 92556 79197 69120
16	4769 60569 02119 12541 41573 72421 73104
17	422 02183 66027 01955 90270 30610 05750
18	32 79726 82562 00369 41424 57714 51675
19	2 24210 73793 98136 68358 30149 99750
20	13493 48140 96325 64865 50042 31785
21	714 89906 11586 20597 06051 71000
22	33 31855 85651 26608 06201 41100
23	1 36382 47971 28826 97603 07800
24	4890 78062 39173 85069 12340
25	153 11291 06528 21356 96500
26	4 16480 68288 64871 59850
27	9781 65015 05256 39540
28	196 75366 66898 10846
29	3 35349 77823 27000
30	4775 42816 18780
31	55 74142 45080
32	51926 09268
33	371 10150
34	1 90995
35	630
36	1

m	$\mathbf{I}_{37}^{(m)}$
1	37 19933 26789 90121 74679 99448 15083 52000 00000
2	155 29081 63497 01573 26717 25036 96755 91680 00000
3	294 04967 60247 90576 42429 43941 97903 93344 00000
4	340 34688 61017 12592 21416 79434 57581 03961 60000
5	272 87755 64837 88559 42867 25947 20190 07078 40000
6	162 49452 81714 86266 69022 19800 34702 86487 55200
7	75 14462 56205 04946 03342 76418 88499 31426 20160
8	27 83394 16827 42388 56461 96749 77336 54300 42624
9	8 44675 24515 35074 58129 30195 98308 42416 39424
10	2 13644 28539 94695 98184 80748 64116 06169 52320
11	45642 99780 73162 75533 46245 69743 65078 31040
12	8323 89934 19335 30716 42888 71682 74522 91072
13	1306 84102 23348 83096 82298 81596 95476 08832
14	177 83531 79126 45508 68461 34540 45647 35680
15	21 09057 81655 96466 89525 05590 66076 45600
16	2 18940 97030 88953 20840 89210 86380 00864
17	19962 39180 79091 82953 91304 74383 80104
18	1602 72349 38259 15254 81555 08332 66050
19	113 51313 39145 33290 02323 43114 42675
20	7 09976 06868 65860 03516 31673 44010
21	39229 84761 13429 06359 67903 87785
22	1914 36716 95031 78487 29302 50600
23	82 41625 12617 64379 19912 21900
24	3 12450 58217 39085 60091 52040
25	10402 84540 74189 53919 86340
26	303 04595 64919 56734 51100
27	7 68620 08830 54101 83290
28	16864 78215 13588 29996
29	317 47958 68535 82846
30	5 07265 19206 03080
31	6782 11944 41660
32	74 43481 78728
33	65285 74668
34	439 85970
35	2 13675
36	666
37	1

Stirling I Numbers $n = 38$

m	$I_{38}^{(m)}$
1	1376 37530 91226 34504 63159 79581 58090 24000 00000
2	5782 95953 76179 48332 63218 25815 95052 44160 00000
3	11035 12882 92669 52900 96606 50890 19201 45408 00000
4	12886 88446 17881 56488 34850 83021 28402 39923 20000
5	10436 81647 60018 89291 07505 39481 04613 65862 40000
6	6285 17509 88287 80426 96688 58560 04196 07117 82400
7	2942 84567 61301 69269 92704 47299 09177 49257 01120
8	1105 00046 78819 73322 92435 56160 49951 40541 97248
9	340 36378 23895 40148 07246 14001 14748 23707 01312
10	87 49513 80493 38825 90967 17895 70602 70688 75264
11	19 02435 20427 01717 92922 91839 44631 14067 00800
12	3 53627 27345 88569 12041 33128 22005 22426 00704
13	56677 01716 83242 05298 87944 90770 07138 17856
14	7886 74778 51027 66918 15368 59593 84428 28992
15	958 18671 00397 14783 80888 41394 90476 22880
16	102 09873 71798 87735 60638 06392 62136 77568
17	9 57549 46720 15350 90135 67486 38580 64712
18	79263 16107 94680 47382 08842 82692 23954
19	5802 70944 86636 46985 67522 03566 45025
20	376 20427 93285 70111 32427 15031 71045
21	21 61480 43030 62735 38824 44116 92055
22	1 10061 43288 29605 10389 52096 59985
23	4963 76846 61884 60517 66054 60900
24	198 02296 66661 10546 43298 47380
25	6 97355 86224 84098 55126 46620
26	21615 54579 76213 53096 77040
27	587 43538 91649 58502 32830
28	13 92617 02790 56868 93142
29	28611 52686 49413 95298
30	505 16770 79158 96806
31	7 58203 61149 44500
32	9536 20770 54596
33	98 59054 41444
34	81560 55558
35	518 91945
36	2 38317
37	703
38	1

m	$I_{39}^{(m)}$
1	52302 26174 66601 11176 00072 24100 07429 12000 00000
2	2 21128 83773 86046 71144 65453 60587 70083 02080 00000
3	4 25117 85504 97621 58569 34265 59643 24707 69664 00000
4	5 00736 73837 72168 99458 20938 05698 98492 62489 60000
5	4 09485 91054 98599 49549 20055 83301 03721 42694 40000
6	2 49273 47023 14955 45515 81671 64762 64064 36339 71200
7	1 18113 31079 17752 12684 19458 55925 52940 78884 24960
8	44932 86345 56451 55541 05255 81398 07330 89851 96544
9	14038 82419 86844 98949 67788 88204 10384 41408 47104
10	3665 17902 82644 15532 63998 94037 97651 09879 61344
11	810 42051 56720 04107 22038 07794 66586 05235 05664
12	153 40271 59570 67344 50493 50711 80829 66255 27552
13	25 07353 92585 51767 13398 75034 71267 93676 79232
14	3 56373 43300 22293 48188 71951 55336 15413 19552
15	44297 84276 66119 28702 89128 32600 22524 98432
16	4837 93872 28754 48736 85134 84314 51673 70464
17	465 96753 47164 71069 85793 70875 28201 36624
18	39 69549 58822 13208 90655 03513 80885 74964
19	2 99766 12012 86866 32837 74680 18217 34904
20	20098 47206 31493 11215 99753 74771 44735
21	1197 56684 28449 54056 07755 91474 69135
22	63 43814 87985 87729 33626 23787 71485
23	2 98684 63459 81220 10060 62171 74185
24	12488 64119 95006 61282 11396 61340
25	463 01819 43205 06291 38104 18940
26	15 18746 60255 80212 72803 74140
27	43938 09058 58897 76185 24580
28	1116 62985 97691 19521 72226
29	24 79855 04877 34599 14466
30	47807 89976 57454 73926
31	793 28508 02837 87806
32	11 20579 50430 19148
33	13282 64838 29468
34	129 58355 52648
35	1 01279 49468
36	609 47991
37	2 65031
38	741
39	1

Stirling I Numbers $n = 40$

m	$\mathbf{I}_{40}^{(m)}$
1	20 39788 20811 97443 35864 02817 39902 89735 68000 00000
2	86 76326 93355 22422 85817 52762 87020 40666 93120 00000
3	168 00725 18467 93288 55349 01811 86674 33683 18976 00000
4	199 53850 65176 12212 37439 50849 81903 65920 06758 40000
5	164 70687 24982 17549 31877 03115 54439 43628 27571 20000
6	101 31151 24957 81862 24666 05250 09044 02231 59943 16800
7	48 55692 59111 07288 40199 40555 45858 28755 12825 44640
8	18 70494 98556 19362 78785 24435 30450 38845 83110 90176
9	5 92447 00720 43406 14578 49022 21358 12323 04782 33600
10	1 56980 80630 09967 04722 63747 55685 18777 26713 39520
11	35271 57913 94725 75714 23483 98029 94507 14046 82240
12	6793 12643 79976 30542 91284 85555 18942 89190 80192
13	1131 27074 70405 86262 73044 77065 60279 19650 17600
14	164 05917 81294 21212 92758 81145 29377 94791 41760
15	20 83989 30090 00945 67601 47956 26744 93887 58400
16	2 32977 45295 87544 29440 09387 20866 37799 46528
17	23010 67257 68178 20461 31089 48450 51526 98800
18	2014 09187 41227 86217 21340 07913 82745 60220
19	156 60428 27324 00995 71327 16040 91362 36220
20	10 83606 53059 15097 70261 65076 34303 79569
21	66803 57893 41025 19403 02234 42284 41000
22	3671 65464 59898 75500 19179 19195 57050
23	179 92515 62918 55313 25990 48485 64700
24	7 85741 64137 86478 00063 06639 66445
25	30546 35077 80004 06645 97460 00000
26	1055 32936 93181 34587 77450 10400
27	32 32332 13540 77225 44028 32760
28	87486 65511 68854 37532 41394
29	2083 77332 87907 68888 36400
30	43 44363 13963 75333 97580
31	78746 01789 68131 98360
32	1230 31108 69615 34578
33	16 38602 79123 68400
34	18336 40703 82740
35	169 08255 81900
36	1 25049 21117
37	712 84200
38	2 93930
39	780
40	1

Table of Stirling II Numbers $\Pi_n^{(m)}$ $n=1(1)40$

Stirling II Numbers $n=1(1)21$

m	$\Pi_1^{(m)}$	$\Pi_2^{(m)}$	$\Pi_3^{(m)}$	$\Pi_4^{(m)}$	$\Pi_5^{(m)}$	$\Pi_6^{(m)}$	$\Pi_7^{(m)}$	$\Pi_8^{(m)}$	$\Pi_9^{(m)}$	$\Pi_{10}^{(m)}$	$\Pi_{11}^{(m)}$	$\Pi_{12}^{(m)}$
1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1023	2047
3			1	6	25	90	301	966	3025	9330	28501	86526
4				1	10	65	350	1701	7770	34105	1 45750	6 11501
5					1	15	140	1050	6951	42525	2 46730	13 79400
6						1	21	266	2646	22827	1 79487	13 23652
7							1	28	462	5880	63987	6 27396
8								1	36	750	11880	1 59027
9									1	45	1155	22275
10										1	55	1705
11											1	66
12												1

m	$\Pi_{13}^{(m)}$	$\Pi_{14}^{(m)}$	$\Pi_{15}^{(m)}$	$\Pi_{16}^{(m)}$	$\Pi_{17}^{(m)}$
1	1	1	1	1	1
2	4095	8191	16383	32767	65535
3	2 61625	7 88970	23 75101	71 41686	214 57825
4	25 32530	103 91745	423 55950	1717 98901	6943 37290
5	75 08501	400 75035	2107 66920	10961 90550	56527 51651
6	93 21312	634 36373	4206 93273	27349 26558	1 75057 49898
7	57 15424	493 29280	4087 41333	32818 82604	2 57081 04786
8	18 99612	209 12320	2166 27840	21417 64053	2 04159 95028
9	3 59502	51 35130	671 28490	8207 84250	95288 22303
10	39325	7 52752	126 62650	1937 54990	27583 34150
11	2431	66066	14 79478	289 36908	5120 60978
12	78	3367	1 06470	27 57118	620 22324
13	1	91	4550	1 65620	49 10178
14		1	105	6020	2 49900
15			1	120	7820
16				1	136
17					1

m	$\Pi_{18}^{(m)}$	$\Pi_{19}^{(m)}$	$\Pi_{20}^{(m)}$	$\Pi_{21}^{(m)}$
1	1	1	1	1
2	1 31071	2 62143	5 24287	10 48575
3	644 39010	1934 48101	5806 06446	17423 43625
4	27988 06985	1 12596 66950	4 52321 15901	18 15090 70050
5	2 89580 95545	14 75892 84710	74 92060 90500	379 12625 68401
6	11 06872 51039	69 30816 01779	430 60788 95384	2658 56794 62804
7	19 74624 83400	149 29246 34839	1114 35540 45652	8231 09572 14948
8	18 90360 65010	170 97510 03480	1517 09326 62679	13251 10153 47084
9	10 61753 95755	114 46146 26805	1201 12826 44725	12327 24764 65204
10	3 71121 63803	47 72970 33785	591 75849 64655	7118 71322 91275
11	83910 04908	12 94132 17791	190 08424 29486	2682 68516 89001
12	12563 28866	2 34669 51300	41 10166 33391	683 30420 30178
13	1258 54638	28924 39160	6 10686 60380	120 49092 18331
14	84 08778	2435 77530	63025 24580	14 93040 04500
15	3 67200	139 16778	4523 29200	1 30874 62580
16	9996	5 27136	223 50954	8099 44464
17	153	12597	7 41285	349 52799
18	1	171	15675	10 23435
19		1	190	19285
20			1	210
21				1

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Stirling II Numbers $n = 22(1)26$

m	$\Pi_{22}^{(m)}$	$\Pi_{23}^{(m)}$	$\Pi_{24}^{(m)}$
1	1	1	1
2	20 97151	41 94303	83 88607
3	52280 79450	1 56863 35501	4 70632 00806
4	72 77786 23825	291 63425 74750	1168 10566 34501
5	1913 78219 12055	9641 68881 84100	48500 07834 95250
6	16330 53393 45225	99896 98579 83405	6 09023 60360 84530
7	60276 23799 67440	4 38264 19991 17305	31 67746 38518 04540
8	1 14239 90799 91620	9 74195 50199 00400	82 31828 21583 20505
9	1 24196 33035 33920	12 32006 88117 96900	120 62257 43260 72500
10	83514 37993 77954	9 59340 12973 13460	108 25408 17849 31500
11	36628 25008 70286	4 86425 13089 51100	63 10016 56957 75560
12	10882 33560 51137	1 67216 27734 83930	24 93020 45907 58260
13	2249 68618 68481	40128 25603 41390	6 88883 60579 22000
14	329 51652 81331	6862 91758 07115	1 36209 10216 41000
15	34 56159 43200	847 94044 29331	19582 02422 47080
16	2 60465 74004	76 23611 27264	2067 71824 65555
17	14041 42047	4 99169 88803	161 09499 36915
18	533 74629	23648 85369	9 24849 25445
19	13 89850	797 81779	38807 39170
20	23485	18 59550	1169 72779
21	231	28336	24 54606
22	1	253	33902
23		1	276
24			1

m	$\Pi_{25}^{(m)}$	$\Pi_{26}^{(m)}$
1	1	1
2	167 77215	335 54431
3	14 11979 91025	42 36107 50290
4	4677 12897 38810	18722 63569 46265
5	2 43668 49741 10751	12 23019 61602 92565
6	37 02641 70000 02430	224 59518 69741 25331
7	227 83248 29987 16310	1631 85379 79910 16600
8	690 22372 11183 68580	5749 62225 19456 64950
9	1167 92145 10929 73005	11201 51678 09551 25625
10	1203 16339 21753 87500	13199 55537 28468 48005
11	802 35590 44384 62660	10029 07834 09984 76760
12	362 26262 07848 74680	5149 50735 38569 58820
13	114 48507 33437 44260	1850 56857 42535 50060
14	25 95811 03608 96000	477 89861 83962 88260
15	4 29939 46553 47200	90 44903 01911 04000
16	52665 51616 95960	12 72587 72424 82560
17	4806 33313 93110	1 34373 17953 78830
18	327 56785 94925	10702 55461 01760
19	16 62189 69675	643 38390 18750
20	62201 94750	29 06228 64675
21	1685 19505	97591 04355
22	32 00450	2389 29405
23	40250	41 26200
24	300	47450
25	1	325
26		1

m	$\mathrm{II}^{(m)}_{27}$	$\mathrm{II}^{(m)}_{28}$
1	1	1
2	671 08863	1342 17727
3	127 08658 05301	381 26645 24766
4	74932 90385 35350	2 99858 70199 46701
5	61 33820 71584 09090	307 44036 48305 80800
6	1359 80131 80050 44551	8220 14611 51886 76396
7	11647 57177 29112 41531	82892 80372 83837 35268
8	47628 83181 35563 36200	3 92678 22628 13619 31131
9	1 06563 27328 05417 95575	10 06698 29133 84324 96375
10	1 43197 07050 94236 05675	15 38533 97837 47778 52325
11	1 23519 41712 38300 92365	15 01910 65887 15546 21690
12	71823 16658 72819 82600	9 85397 41617 12138 83565
13	29206 89881 91531 09600	4 51512 85123 62724 07400
14	8541 14923 18015 85700	1 48782 98806 43753 09400
15	1834 63407 12628 48260	36060 66030 07443 09600
16	294 06306 60708 24960	6539 64312 83960 47620
17	35 56931 77639 22670	898 74146 80575 10350
18	3 27019 16252 10510	94 43276 70177 11850
19	22926 84874 58010	7 62629 28869 12700
20	1224 62963 12250	47419 44137 03010
21	49 55640 56130	2265 31414 90980
22	1 50155 51265	82 59061 83960
23	3338 32005	2 26936 87380
24	52 65000	4601 92005
25	55575	66 54375
26	351	64701
27	1	378
28		1

m	$\mathrm{II}^{(m)}_{29}$	$\mathrm{II}^{(m)}_{30}$
1	1	1
2	2684 35455	5368 70911
3	1143 81277 92025	3431 46518 11530
4	11 99816 07443 11570	48 00408 11050 38305
5	1540 20041 11728 50701	7713 00021 66085 65075
6	49628 31705 59626 39176	2 99310 10274 69486 85757
7	5 88469 77221 38748 23272	41 68916 72255 30864 02080
8	32 24318 61397 92791 84316	263 83018 68404 81082 97800
9	94 52962 84832 72543 98506	883 00984 24892 45687 70870
10	163 92038 07508 62110 19625	1733 73343 59918 93645 94756
11	180 59551 22596 18786 90915	2150 47101 56066 68766 19690
12	133 26679 65292 61212 24470	1779 79707 06107 53333 84555
13	68 55064 48224 27551 79765	1024 42517 92208 19385 61415
14	25 34474 68413 75267 39000	423 37710 06016 81295 25765
15	6 89692 89257 55399 53400	128 79868 07277 06260 40000
16	1 40694 95035 50810 71520	29 40812 09825 68370 97720
17	21818 24808 53737 23570	5 11605 16780 64343 72210
18	2598 53127 43763 23650	68591 81102 41475 49270
19	239 33233 18690 53150	7145 84557 98883 33500
20	17 11018 11609 72900	581 53595 50885 11150
21	94991 03850 13590	37 05829 92462 58290
22	4082 30775 38100	1 84801 80908 51790
23	134 78609 93700	7182 38803 93200
24	3 37382 95500	215 75800 85700
25	6265 51380	4 94020 80000
26	83 36601	8433 03006
27	74907	103 59090
28	406	86275
29	1	435
30		1

Stirling **II** Numbers $n=31(1)32$

m	$\mathbf{II}_{31}^{(m)}$	$\mathbf{II}_{32}^{(m)}$
1	1	1
2	10737 41823	21474 83647
3	10294 44923 05501	30883 45506 58326
4	192 05063 90719 64750	768 30550 07801 64501
5	38613 00516 41478 63680	1 93257 07645 98112 83150
6	18 03573 61669 83006 79617	108 60054 70535 39519 41382
7	294 81727 16061 85535 00317	2081 75663 74102 81751 81836
8	2152 33066 19493 79527 84480	17513 46256 72012 21757 76157
9	8210 91876 92436 92272 35630	76050 59958 51426 09979 05150
10	18220 34420 24081 82147 18430	1 90414 36079 33255 13744 19930
11	25388 91460 76652 50074 11346	2 97498 40488 67259 32962 43236
12	23508 03586 29357 08772 34350	3 07485 34496 28937 55342 23546
13	15097 32440 04814 05346 82950	2 19773 25306 91939 78281 12700
14	6951 70458 76443 57519 22125	1 12421 18862 75024 10615 92700
15	2355 35731 15172 75201 25765	42282 06426 04034 85538 08600
16	599 32861 64488 00196 03520	11944 61517 46980 78337 82085
17	116 38099 95096 62214 25290	2577 80560 81130 57838 33450
18	17 46257 76624 10902 59070	430 70739 74330 58460 88550
19	2 04362 87704 20258 85770	56 29152 43003 95820 88700
20	18776 56468 16585 56500	5 79894 17067 51970 15770
21	1359 76023 92599 35240	47331 52970 61171 96540
22	77 71469 72449 97670	3069 48357 86498 83980
23	3 49996 73398 95390	158 21394 60625 91640
24	12360 58024 50000	6 46650 65986 95390
25	339 26320 85700	20842 16045 92500
26	7 13279 58156	524 71589 97756
27	11229 98436	10 16489 15928
28	127 74790	14806 92556
29	98890	156 42600
30	465	1 12840
31	1	496
32		1

m	$\mathrm{II}_{33}^{(m)}$	$\mathrm{II}_{34}^{(m)}$
1	1	1
2	42949 67295	85899 34591
3	92650 57994 58625	2 77952 16933 43170
4	3073 53083 76713 16330	12295 04985 64847 23945
5	9 67053 68779 98365 80251	48 38341 96983 68542 17585
6	653 53585 30858 35229 31442	3930 88565 53930 09741 68903
7	14680 89700 89255 11782 14234	1 03419 81491 55644 17704 31080
8	1 42189 45717 50200 55813 91092	11 52196 55440 90859 58293 42970
9	7 01968 85883 34847 11569 22507	64 59909 18667 63824 59936 93655
10	19 80194 20751 83977 47421 04450	205 03910 93401 74621 85779 67007
11	34 62896 81454 73107 76330 95526	400 72059 16753 88162 87061 55236
12	39 87322 54444 14509 97069 25788	513 10767 34784 47227 41162 04982
13	31 64537 63486 24154 72996 88646	451 26311 79765 28521 46028 78186
14	17 93669 89385 42277 26904 10500	282 75916 14882 16036 49654 35646
15	7 46652 15253 35546 93687 21700	129 93452 18185 75481 32212 36000
16	2 33395 90705 55727 38943 21960	44 80986 66542 27185 16778 73060
17	55767 31051 26200 61589 50735	11 81440 18577 01137 85964 84455
18	10330 53876 19081 10134 27350	2 41717 00822 69660 44006 43035
19	1500 24635 91405 79057 73850	38835 21958 55791 12231 30500
20	172 27035 84354 35224 04100	4945 65352 78492 83538 55850
21	15 73856 29450 36581 43110	502 78018 02812 03434 09410
22	1 14860 16843 64146 44100	41 00780 00010 47803 13310
23	6708 40433 80894 91700	2 69153 46821 24729 53200
24	313 41010 44312 81000	14230 24684 44402 35700
25	11 67704 67135 07890	605 33627 22689 78250
26	34484 77385 34156	20 64308 79153 95946
27	799 16797 27812	56062 30911 85080
28	14 31083 07496	1199 87123 37700
29	19343 27956	19 92038 18220
30	190 27800	25051 61956
31	1 28216	230 02496
32	528	1 45112
33	1	561
34		1

Stirling II Numbers $n = 35(1)36$

m	$\Pi_{35}^{(m)}$	$\Pi_{36}^{(m)}$
1	1	1
2	1 71798 69183	3 43597 38367
3	8 33857 36699 64101	25 01573 81897 61486
4	49182 97894 76322 38950	1 96740 25436 41989 19901
5	242 04004 89904 07558 11870	1210 69207 47415 14112 98300
6	23633 69735 20564 26992 31003	1 42044 22416 13289 69511 97888
7	7 27869 59006 43439 33671 86463	51 18720 82780 24639 62695 36244
8	93 20992 25018 82520 84051 74840	752 95807 59157 03606 06085 85183
9	592 91379 23449 65280 97725 85865	5429 43405 36065 70049 63584 47625
10	2114 99018 52685 10043 17733 63725	21742 81564 50300 65712 75062 23115
11	4612 96561 77694 44413 43456 74603	52857 61198 07323 98590 95757 84358
12	6558 01267 34167 54891 81006 15020	83309 11769 87705 03115 15530 54843
13	6379 52820 71733 18006 39536 21400	89491 87936 66698 88974 94976 93220
14	4409 89137 88115 53032 41189 77230	68118 00751 05350 60460 16193 02620
15	2231 77698 87668 48256 32839 75646	37886 54621 03142 76877 33786 11920
16	846 89238 82862 10444 00672 04960	15782 05520 13462 15360 43592 55006
17	245 65469 82351 46528 78181 08795	5023 02225 82837 01433 29750 54475
18	55 32346 33385 55025 78080 59085	1241 47703 83291 36992 83631 72325
19	9 79586 18035 29691 76401 22535	241 44483 76056 19169 29703 87250
20	1 37748 29014 25647 83002 47500	37 34551 98320 42648 36450 72535
21	15504 03731 37545 55654 53460	4 63333 07373 14104 51747 70160
22	1404 95178 03042 55103 02230	46412 97648 04481 67921 02520
23	102 91309 76899 16582 36910	3771 95302 71723 36497 51160
24	6 10679 39247 90386 10000	249 47615 18848 85848 76910
25	29363 65365 11646 91950	13 44770 73375 81559 08750
26	1142 05655 80692 72846	59057 12416 09657 85946
27	35 77991 13773 93106	2108 11416 52588 86708
28	89658 70366 40680	60 88434 84033 32146
29	1777 56230 66080	1 41208 01055 57000
30	27 43586 76900	2600 63833 73080
31	32182 39332	37 41240 96192
32	276 46080	41029 13892
33	1 63625	330 45705
34	595	1 83855
35	1	630
36		1

m	$\mathrm{II}_{37}^{(m)}$
1	1
2	6 87194 76735
3	75 04724 89290 22825
4	7 86986 03319 49854 41090
5	6055 42777 62512 12554 11401
6	8 53476 03704 27153 31184 85628
7	359 73090 01877 85767 08379 51596
8	6074 85181 56036 53488 11382 17708
9	49617 86455 83748 34052 78346 13808
10	2 22857 59050 39072 27177 14206 78775
11	6 03176 54743 30864 50213 28398 51053
12	10 52567 02436 59784 35972 82124 42474
13	12 46703 54946 54790 59789 50230 66703
14	10 43143 98451 41607 35417 21679 29900
15	6 36416 20066 52492 13620 22984 81420
16	2 90399 42943 18537 22644 31266 92016
17	1 01173 43359 21691 39726 49351 81081
18	27369 60894 82081 67304 35121 56325
19	5828 92895 28359 01209 48005 30075
20	988 35523 42464 72136 58718 37950
21	134 64546 53156 38843 23152 45895
22	14 84418 55630 12701 46010 25600
23	1 33167 89610 54119 07363 79200
24	9759 38067 24095 96867 97000
25	585 66883 53244 24825 95660
26	28 80255 96194 32663 43346
27	1 15976 20662 29557 27062
28	3812 87592 05521 86796
29	101 83467 14644 85146
30	2 19227 16067 49400
31	3760 42303 55032
32	50 54173 40736
33	51934 22157
34	392 96775
35	2 05905
36	666
37	1

Stirling II Numbers $n = 38$

m	$\Pi_{38}^{(m)}$
1	1
2	13 74389 53471
3	225 14181 55065 45210
4	31 48019 18002 88707 87185
5	30285 00874 15880 12624 98095
6	51 26911 65003 25431 99663 25169
7	2526 65106 16849 27522 89841 46800
8	48958 54542 50170 13671 99436 93260
9	4 52635 63284 09771 59963 16497 41980
10	22 78193 76959 74471 05824 20414 01558
11	68 57799 61226 78581 79523 26590 40358
12	132 33980 83982 48276 81887 13891 60741
13	172 59713 16741 72062 13236 35123 09613
14	158 50719 33266 37293 55630 53740 85303
15	105 89386 99449 28989 39720 66451 51200
16	52 82807 07157 49087 75929 23255 53676
17	20 10347 80049 87290 97994 70247 70393
18	5 93826 39465 99161 51204 81539 94931
19	1 38119 25905 20902 90284 47222 27750
20	25596 03363 77653 43941 22372 89075
21	3815 91000 58748 87844 44920 01745
22	461 21754 77019 18275 35378 09095
23	45 47280 16672 57440 15377 47200
24	3 67393 03224 32422 32195 07200
25	24401 10155 55202 17516 88500
26	1334 53538 54296 74075 22656
27	60 11613 54076 30709 74020
28	2 22736 73239 84169 57350
29	6766 08139 30222 56030
30	167 60281 96669 67146
31	3 35800 27477 55392
32	5377 75852 58584
33	67 68002 71917
34	65295 12507
35	465 03450
36	2 29881
37	703
38	1

m	$\mathrm{II}_{39}^{(m)}$
1	1
2	27 48779 06943
3	675 42558 39585 89101
4	125 92301 86193 09896 93950
5	1 51456 52389 97403 51832 77660
6	307 91754 90893 68472 10604 49109
7	17737 82654 82948 18092 28553 52769
8	3 94195 01446 18210 36898 85336 92880
9	41 22679 24099 38114 53340 47913 71080
10	232 34573 32881 54482 18205 20637 57560
11	777 13989 50454 38870 80580 12908 45496
12	1656 65569 69016 57903 62168 93289 69250
13	2376 10252 01624 85084 53959 70491 85710
14	2391 69783 82470 94171 92063 87495 03855
15	1746 91524 25005 72134 51440 50513 53303
16	951 14300 13969 14393 54588 38540 10016
17	394 58719 68005 33034 41839 17466 50357
18	126 99222 90437 72198 19681 37966 79151
19	32 18092 31664 96316 66609 78763 22181
20	6 50039 93180 73971 69108 94680 09250
21	1 05730 14376 11379 88674 65693 25720
22	13962 69605 53170 89902 23238 01835
23	1507 09198 60488 39398 89059 94695
24	133 64712 94056 35575 88059 20000
25	9 77420 57113 12476 70117 19700
26	59099 02157 66917 43472 77556
27	2957 67104 14357 03238 21196
28	122 48242 04791 87457 79820
29	4 18953 09279 60623 82220
30	11794 16598 30312 70410
31	271 70090 48473 84298
32	5 07888 54760 30080
33	7611 19942 31845
34	89 88036 97155
35	81571 33257
36	547 79166
37	2 55892
38	741
39	1

Stirling II Numbers $n = 40$

m	$\Pi_{40}^{(m)}$
1	1
2	54 97558 13887
3	2026 27702 67536 74246
4	503 69882 87330 79173 64901
5	7 57408 54251 73210 69060 82250
6	1849 01985 97752 08236 15459 72314
7	1 24472 70338 71530 95118 10479 18492
8	31 71297 94224 28631 13283 11248 95809
9	374 98308 18340 61241 16963 16560 32600
10	2364 68412 52914 82936 35392 54289 46680
11	8780 88457 87879 82061 04586 62630 58016
12	20657 00825 78653 33714 26607 32384 76496
13	32545 98845 90139 64002 63645 09683 83480
14	35859 87225 56218 03491 42853 95422 39680
15	28595 42647 57556 76189 63671 45198 03400
16	16965 20326 48512 02431 24854 67155 13559
17	7659 12534 70059 75978 65854 35470 66085
18	2680 44731 95884 32601 96104 00868 75075
19	738 42976 92072 02214 85267 34468 00590
20	162 18890 95279 75750 48788 72365 07181
21	28 70372 95079 12949 31276 74238 49370
22	4 12909 45697 81139 66523 76929 66090
23	48625 81173 44403 96076 71616 79820
24	4714 62309 17840 93220 02480 74695
25	378 00227 21884 47493 40989 12500
26	25 13995 13212 52330 00409 36156
27	1 38956 13969 54557 30904 49848
28	6387 17881 48529 52056 56156
29	243 97881 73900 45548 64200
30	7 72778 07228 70004 94520
31	20216 89403 33001 83648
32	434 22524 00803 46858
33	7 59058 12856 80965
34	10667 13199 35115
35	118 43033 61150
36	1 01291 83233
37	642 47170
38	2 84050
39	780
40	1

Table of Stirling I′ Numbers $I'^{(m)}_n$ $n=1(1)30$

Stirling I′ Numbers $n=1(1)16$

m	$I'^{(m)}_1$	$I'^{(m)}_2$	$I'^{(m)}_3$	$I'^{(m)}_4$	$I'^{(m)}_5$	$I'^{(m)}_6$	$I'^{(m)}_7$	$I'^{(m)}_8$	$I'^{(m)}_9$
1	1	2	6	24	120	720	5040	40320	3 62880
2		3	20	130	924	7308	64224	6 23376	66 36960
3			15	210	2380	26432	3 03660	36 78840	473 24376
4				105	2520	44100	7 05320	110 98780	1773 31440
5					945	34650	8 66250	188 58840	3894 49060
6						10395	5 40540	182 88270	5200 59540
7							1 35135	94 59450	4162 15800
8								20 27025	1837 83600
9									344 59425

m	$I'^{(m)}_{10}$	$I'^{(m)}_{11}$	$I'^{(m)}_{12}$	$I'^{(m)}_{13}$
1	36 28800	399 16800	4790 01600	62270 20800
2	769 98240	9675 24480	1 30967 36640	19 00603 35360
3	6475 36032	94189 45536	14 54105 80224	237 76097 52960
4	29205 25608	4 99528 62960	89 05771 27440	1657 58033 22624
5	79349 27000	16 28317 89120	340 45544 33280	7301 72365 32240
6	1 36426 29000	34 52408 96000	863 72356 47040	21675 22214 45760
7	1 49804 05440	48 65915 85480	1497 29846 66640	44859 41859 59920
8	1 01999 89800	45 32471 14320	1785 69352 96200	65659 83992 56800
9	39283 74450	26 84389 20750	1443 37207 01400	67810 37759 49600
10	6547 29075	9 16620 70500	756 21208 16250	48390 85133 88300
11		1 37493 10575	231 90503 83650	22726 69375 97700
12			31 62341 43225	6324 68286 45000
13				790 58535 80625

m	$I'^{(m)}_{14}$	$I'^{(m)}_{15}$	$I'^{(m)}_{16}$
1	8 71782 91200	130 76743 68000	2092 27898 88000
2	294 43103 42400	4850 38181 37600	84679 53725 95200
3	4108 27214 13120	74845 95398 43840	14 34534 04436 65920
4	32220 80222 84928	6 53923 33865 64864	138 46616 56017 65376
5	1 61267 47173 87552	36 76277 20537 77120	866 04010 88068 39680
6	5 50561 97015 82000	142 36588 83793 91040	3761 70186 90965 31360
7	13 30692 81481 13600	395 06350 48436 07600	11823 44665 09059 70080
8	23 20904 42895 51120	803 35139 36286 23840	27563 54266 48613 23120
9	29 36344 78545 40800	1209 16731 93141 14160	48300 44911 06257 12000
10	26 72628 26547 71700	1346 15353 22235 00000	63883 02128 84403 54000
11	17 06821 08236 64000	1094 86233 69608 92500	63466 41259 87942 05000
12	7 26284 41560 67500	632 60742 94730 19000	46641 68369 37156 10500
13	1 84996 97378 66250	246 04597 51362 11250	24602 29532 90584 47000
14	21345 80466 76875	57 77597 79672 07500	8810 83663 99991 43750
15		6 19028 33536 29375	1918 98783 96251 06250
16			191 89878 39625 10625

Stirling I′ Numbers $n = 17(1)20$

m	$\mathbf{I}'^{(m)}_{17}$	$\mathbf{I}'^{(m)}_{18}$
1	35568 74280 96000	6 40237 37057 28000
2	15 61892 69246 97600	303 51767 27030 78400
3	288 65058 05089 61280	6085 39014 86731 77600
4	3056 23012 09086 25920	70242 49472 97693 31200
5	21094 63176 25807 06176	5 31318 96143 67653 06112
6	1 01810 32351 38741 62880	28 26813 97135 84619 88288
7	3 58458 41596 00575 33120	110 46449 74737 43607 04000
8	9 45287 74357 84150 36800	225 93653 98846 18142 48000
9	18 96599 79438 71758 78000	738 89075 98710 53637 84800
10	29 16770 23037 57177 16000	1299 60990 66859 81270 38000
11	34 38434 71495 53330 93000	1779 45738 46926 94226 52000
12	30 83026 69619 02748 34000	1891 22380 92322 26298 83000
13	20 66075 39166 04476 67500	1544 73062 63552 16750 45000
14	10 02393 95907 17277 22500	951 22549 87269 74370 90000
15	3 32624 55886 83517 50000	427 20592 57408 25431 20000
16	67548 37195 48037 40000	132 05706 71716 41311 70000
17	6332 65987 07628 50625	25 11955 08206 92640 81250
18		2 21643 09547 66997 71875

m	$\mathbf{I}'^{(m)}_{19}$	$\mathbf{I}'^{(m)}_{20}$
1	121 64510 04088 32000	2432 90200 81766 40000
2	6198 40092 81761 28000	1 32720 96660 02841 60000
3	1 34167 06424 89013 76000	30 88040 23389 57050 88000
4	16 79213 46732 57351 93600	417 07752 22621 66411 00800
5	138 35913 49183 02966 58176	3723 63047 01974 47644 42624
6	805 95190 38708 54550 65600	23607 77596 97289 37930 94400
7	3468 31592 96832 05673 07200	1 11130 96367 24055 65816 92800
8	11346 42697 13174 05487 52000	3 99998 05832 70165 01335 98400
9	28750 33709 34031 38068 85600	11 22709 39381 21752 19578 52800
10	57078 01866 35969 77430 38400	24 89022 31695 30033 49477 96000
11	89292 95144 99815 89410 10000	43 91129 10340 73570 05214 52000
12	1 10120 43581 77476 15760 50000	61 81815 00529 96053 60288 60000
13	1 06514 58750 32107 34527 68000	69 32320 74627 06672 09221 76000
14	79870 59600 26301 15883 20000	61 50711 05569 27480 63559 04000
15	45488 23700 74373 93469 30000	42 62200 32234 22953 17985 00000
16	19014 94175 90238 69258 60000	22 57611 25682 61441 95476 50000
17	5501 18162 97316 88337 93750	8 82580 44199 52000 73475 35000
18	984 09534 39165 46987 12500	2 39955 24802 49847 07027 31250
19	82 00794 53263 78915 59375	40511 92499 12311 84303 31250
20		3198 30986 77287 77708 15625

m	$\mathbf{I}'^{(m)}_{21}$
1	51090 94217 17094 40000
2	29 73385 10938 61376 00000
3	740 77507 61140 77527 04000
4	10750 99019 04269 63085 31200
5	1 03517 69981 14903 51385 85600
6	7 10616 56743 80860 24959 62624
7	36 37945 97033 76316 01192 54400
8	143 11612 61598 38178 80281 53600
9	441 58516 11203 65599 06520 84800
10	1083 51951 32295 53570 71694 64000
11	2132 84694 03117 11709 95466 88000
12	3383 34211 47862 27956 96099 84000
13	4327 66479 80181 89947 93841 88000
14	4448 23081 26675 61192 74547 20000
15	3644 51898 23122 65183 54041 40000
16	2347 13216 85006 38224 84614 00000
17	1161 87092 85638 97379 51218 45000
18	426 56356 22076 70216 59101 17500
19	109 38219 74763 24197 61894 37500
20	17 48409 39435 83984 80458 75000
21	1 31130 70457 68798 86034 40625

m	$\mathbf{I}'^{(m)}_{22}$
1	11 24000 72777 76076 80000
2	695 62949 18583 04819 20000
3	18492 21425 29905 33672 96000
4	2 87294 13166 35260 15308 80000
5	29 70985 94004 98501 76250 36800
6	219 81625 21573 85621 61328 02048
7	1217 59751 05772 00935 32260 76672
8	5205 37199 00332 20349 62748 32000
9	17541 03861 84061 13336 04071 52000
10	47278 24490 48474 94263 24680 12800
11	1 02923 72651 33204 88981 49168 64000
12	1 82034 23881 82320 09008 21701 76000
13	2 62174 23503 53502 08766 58018 48000
14	3 07156 34637 40012 89923 93617 80000
15	2 91338 99261 92737 49546 29189 60000
16	2 21691 09258 00774 26110 30249 80000
17	1 33342 11768 84523 52965 61633 10000
18	61948 94514 00911 36248 02465 37500
19	21437 83038 73597 76568 39822 00000
20	5201 51794 82162 35479 36478 12500
21	789 40684 15528 16913 92712 56250
22	56 38620 29680 58350 99479 46875

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Stirling I′ Numbers $n = 23(1)24$

m	$I'^{(m)}_{23}$
1	258 52016 73888 49766 40000
2	16964 86797 92659 41504 00000
3	4 79696 09362 12209 62304 00000
4	79 50444 99382 94302 73525 76000
5	879 73561 93626 11571 72097 53600
6	6986 73112 36207 55454 52194 87744
7	41684 99911 93030 10151 14074 82880
8	1 92689 08501 83126 38548 50272 60160
9	7 05138 72886 16193 44255 71415 04000
10	20 74217 07274 41154 43177 20052 73600
11	49 56665 05679 95434 47076 37009 34400
12	96 88570 82127 27849 31650 09593 60000
13	155 47296 58487 53776 22117 90208 40000
14	204 95900 93073 66539 52858 58906 08000
15	221 44327 54275 11764 60398 43873 80000
16	194 95143 23757 53446 74950 58697 20000
17	138 46295 20047 26613 83960 83433 10000
18	78 11642 51314 17395 68545 63939 00000
19	34 18857 79662 54874 25473 33782 37500
20	11 18852 63009 41925 06006 04605 25000
21	2 57609 76596 00692 52911 55199 56250
22	37214 89395 89185 11656 56449 37500
23	2537 37913 35626 25794 76576 09375

m	$I'^{(m)}_{24}$
1	6204 48401 73323 94393 60000
2	4 30584 70366 63697 81760 00000
3	129 13185 00161 26594 99008 00000
4	2276 13809 36116 75833 67403 52000
5	26858 72194 04253 64484 77452 28800
6	2 28127 53554 65176 43761 04479 98976
7	14 60151 90728 77129 68169 88091 18720
8	72 65596 60826 60851 09688 94770 34240
9	287 30490 04415 78234 49734 94004 53120
10	917 18741 45298 92479 85286 18436 60800
11	2390 49992 40448 44022 68621 40110 72000
12	5125 83255 73253 14932 55426 31103 04000
13	9084 91226 62133 38519 35647 92872 00000
14	13335 98308 07764 51682 74130 17235 76000
15	16203 28681 99253 75557 03767 05635 44000
16	16239 39360 43273 43242 78612 00269 00000
17	13336 57537 52192 02423 56456 85212 00000
18	8879 75446 25819 04390 52765 42256 10000
19	4716 81013 01022 35337 48797 04297 75000
20	1951 21548 34894 62370 53613 50667 87500
21	605 64345 42638 75173 92374 31411 75000
22	132 67109 69635 44494 05565 24202 18750
23	18 28604 56225 41323 22761 19171 56250
24	1 19256 81927 74434 12353 99076 40625

m	$I'^{(m)}_{25}$
1	1 55112 10043 33098 59840 00000
2	113 56518 87977 62565 79993 60000
3	3602 81782 04253 37905 80736 00000
4	67347 55842 15784 68002 59522 56000
5	8 44910 94098 70741 69235 00818 43200
6	76 49587 72460 82902 47374 57968 33280
7	523 36662 72786 11489 69858 69706 48576
8	2792 23952 49772 15384 91482 51568 94720
9	11878 70859 52998 89824 60988 29570 82880
10	40952 73870 90300 04287 90718 22998 73280
11	1 15769 05685 01157 77588 86765 49156 48000
12	2 70587 96932 93257 22388 65717 63695 36000
13	5 25797 55847 09301 77720 69746 87076 48000
14	8 51994 02318 56120 27679 71567 84094 88000
15	11 52031 52612 73712 62351 37991 91976 80000
16	12 97707 21697 01087 51992 95162 36177 60000
17	12 12614 72816 14083 72320 37823 04721 00000
18	9 33085 85318 76464 86191 87335 53660 20000
19	5 84652 27748 54180 08304 67186 01815 55000
20	2 93393 12699 80346 99153 06064 18487 50000
21	1 15058 65219 89001 89500 69451 93583 12500
22	33962 46935 64613 04727 05219 58241 12500
23	7094 98570 15460 33412 31342 38566 25000
24	934 97346 31351 56352 85528 75902 50000
25	58 43584 14459 47272 05345 54743 90625

m	$I'^{(m)}_{26}$
1	40 32914 61126 60563 55840 00000
2	3108 14036 46565 82938 75507 20000
3	1 04058 72425 82468 13205 00428 80000
4	20 57560 91101 81103 71343 67498 24000
5	273 67754 98225 95791 17128 10229 76000
6	2633 29458 63345 62969 14897 22389 70880
7	19195 60014 47902 20549 51464 85594 19392
8	1 09415 00302 44422 86862 24260 02089 28768
9	4 98812 23608 94215 77123 84007 58752 38400
10	18 49100 65565 15462 93938 09728 39934 65600
11	56 41984 64012 92481 47563 89413 97587 66080
12	142 95209 96863 93354 99168 41875 75518 08000
13	302 62650 05640 97242 04155 47651 29329 92000
14	537 33871 68460 51460 10616 11273 75683 04000
15	801 61021 97251 93316 01243 82390 42867 20000
16	1004 39288 46699 66805 88117 59325 54330 40000
17	1054 33521 69552 37192 21159 85387 17741 20000
18	922 65124 99800 93589 16026 81819 10391 60000
19	667 80477 74961 48377 57847 98948 40933 00000
20	395 12043 20175 53718 35597 96259 13637 25000
21	187 88781 84305 90048 78072 73741 55248 75000
22	70 03992 71310 19902 28704 09561 35739 75000
23	19 70757 84278 43522 30689 54974 46754 00000
24	3 93467 99906 93782 98493 26686 08968 75000
25	49670 46522 90551 81245 43715 32320 31250
26	2980 22791 37433 10874 72622 91939 21875

Stirling I′ Numbers $n = 27$

m	$I'^{(m)}_{27}$
1	1088 88694 50418 35216 07680 00000
2	88157 14630 15388 18064 77721 60000
3	31 07839 07406 41984 88169 02144 00000
4	648 48589 05829 07155 36460 37811 20000
5	9121 84792 69160 83741 42625 09568 00000
6	93023 10835 70290 80330 24810 43823 00160
7	7 20353 52612 71178 56115 89948 63468 78976
8	43 72760 50775 39052 51999 74645 81238 37440
9	212 87953 36898 52352 39512 89366 29458 50880
10	845 24864 10267 48433 58229 74495 52733 44000
11	2771 70155 94388 93943 35573 68267 88325 72160
12	7576 13395 13320 61785 75827 89009 78018 15040
13	17377 56540 97691 33284 29631 91554 89072 00000
14	33598 60869 64059 48085 90863 57002 00518 40000
15	54896 90639 94210 35820 86257 40231 60559 84000
16	75852 13038 45967 25119 53179 52070 82299 20000
17	88525 30836 98837 71917 98930 22646 99078 80000
18	86987 40454 51545 54380 36213 57076 37843 20000
19	71570 52123 64308 88503 24366 34538 09607 00000
20	48894 55963 76302 96412 98513 79547 10231 50000
21	27401 38777 10627 57055 42522 90032 37642 00000
22	12380 53178 69572 77651 25287 98539 67448 00000
23	4397 62777 23843 07805 10288 62255 42193 75000
24	1182 11292 09268 65264 59140 83027 86137 50000
25	226 00061 67922 01074 66673 90472 05742 18750
26	27 37836 04342 55215 90248 49588 61495 62500
27	1 57952 07942 83954 76360 49014 72778 59375

m	$I'^{(m)}_{28}$
1	30488 83446 11713 86050 15040 00000
2	25 88134 96415 08389 45144 76646 40000
3	958 79886 61097 21191 87013 95968 00000
4	21066 49272 10300 23347 63511 38611 20000
5	3 12650 68215 99677 08697 30735 16134 40000
6	33 70783 55737 01904 14365 25372 61903 05280
7	276 54805 57246 09958 39169 01808 47920 90624
8	1782 58991 18583 58087 84047 60805 64750 74560
9	9237 85699 56260 90576 94455 04435 85087 79520
10	39150 74246 45142 29081 16477 62887 41102 10560
11	1 37444 10761 76944 10323 64530 25009 60248 14080
12	4 03565 58492 00672 73435 44661 33828 87411 00800
13	9 98147 97444 40478 02802 18392 22586 83606 01600
14	20 90023 13835 31783 36178 40314 90832 73206 40000
15	37 16811 63402 47333 24084 39080 83811 65286 08000
16	56 22208 58171 27637 20436 95787 69004 42938 72000
17	72 32607 30519 71418 69650 92828 87583 80632 00000
18	78 98072 08117 67246 83425 81470 87551 61490 00000
19	72 93664 58595 29303 72645 86676 14265 82709 20000
20	56 61858 80108 08756 91062 75366 62004 32409 50000
21	36 62205 47561 72665 66483 69761 39814 97928 00000
22	19 49314 05834 29817 00627 22733 40030 49410 00000
23	8 38907 97796 70792 72817 78830 39754 82087 50000
24	2 84566 77535 88698 26554 40902 09447 44893 75000
25	73221 90396 13914 49641 42366 21995 77743 75000
26	13429 08579 30021 83400 16887 23215 63604 06250
27	1563 72558 63411 15215 96885 24580 50807 81250
28	86 87364 36856 17511 99826 95810 02822 65625

Stirling I′ Numbers $n = 29$

m	$I'^{(m)}_{29}$
1	8 84176 19937 39701 95454 36160 00000
2	785 58713 95836 03099 35847 50592 00000
3	30525 08668 82881 17020 96920 51046 40000
4	7 04809 33078 84718 25264 16811 06534 40000
5	110 12666 77107 29251 57483 10136 06604 80000
6	1252 36764 14402 53761 84127 07664 53273 39520
7	10858 95619 54409 15188 73699 51338 43838 56640
8	74128 96683 29868 49664 35798 54108 56179 46624
9	4 07756 53557 69246 00597 04598 13935 44026 00960
10	18 38766 77948 53321 47008 15441 58283 95216 23040
11	68 87199 15320 61369 36787 59307 27983 52659 60960
12	216 40387 70151 04673 50363 67663 53539 06365 95200
13	574 70255 93392 87181 25742 85196 13044 11697 98400
14	1297 03186 73748 34978 37184 65699 63621 86121 47200
15	2496 93895 21225 02013 91300 14017 09708 55176 64000
16	4109 16889 49244 98699 58939 34215 23907 61891 20000
17	5784 66714 91094 57515 53954 87745 46470 60682 40000
18	6960 11251 77319 78614 41530 17788 56229 37612 00000
19	7140 11623 35509 37876 35369 02909 85419 77362 40000
20	6218 65122 57762 26910 58013 78052 60967 25697 60000
21	4568 79149 55820 89706 19776 11272 89145 86537 50000
22	2805 75976 69801 24133 55546 24739 92273 66900 00000
23	1421 99323 85181 31096 45695 79753 69051 06372 50000
24	584 20687 17294 93531 67354 26089 58518 03025 00000
25	189 62800 00397 38476 38379 13220 66490 99787 50000
26	46 79153 44673 72561 84245 99686 41416 32781 87500
27	8 24604 62586 38814 23887 57486 28787 92653 12500
28	92433 55688 14970 32766 15883 41870 03306 25000
29	4951 79769 00801 98183 90136 61171 60891 40625

m	$I'^{(m)}_{30}$
1	265 25285 98121 91058 63630 84800 00000
2	24627 29594 88976 26840 70357 89312 00000
3	10 01941 56249 18950 43850 48576 52428 80000
4	242 66035 77673 30780 95409 53142 00166 40000
5	3983 94187 46328 74974 13407 16202 46732 80000
6	47687 30082 02844 05469 56356 23020 95736 83200
7	4 36007 65812 77220 82220 81757 24106 96030 61760
8	31 44553 15205 18272 99564 51428 01539 00667 20768
9	183 11649 09157 66351 09933 35073 85672 07808 08192
10	876 14409 28742 80131 56602 81549 16556 30447 36000
11	3490 38637 30765 87633 51829 89954 50699 15033 60000
12	11696 31061 04338 07757 73202 05803 42426 20048 02560
13	33226 47032 68844 57899 96474 20105 96493 58685 31200
14	80484 58034 87072 52864 05882 88517 96637 06236 60800
15	1 66934 71605 78828 27660 53331 07536 26538 17116 92800
16	2 97274 85311 71150 32107 60776 70455 12727 68052 80000
17	4 55116 45802 55619 85895 93134 10192 37398 38385 60000
18	5 99004 64434 15474 98107 87797 60099 26899 19836 80000
19	6 76810 98006 15799 91556 91161 93523 99159 18771 20000
20	6 54579 60550 70310 74559 75757 67160 72964 49940 00000
21	5 39372 13606 79158 30838 89494 66275 05656 11755 00000
22	3 76102 11439 06729 05827 41440 36653 52396 25312 50000
23	2 19843 15628 59092 71960 64586 33657 88886 10170 00000
24	1 06328 60584 31241 05290 91653 09693 61161 98067 50000
25	41787 08307 55385 28435 09603 22753 50487 51875 00000
26	13003 07439 78911 07102 44382 09889 34902 91315 62500
27	3082 10452 06566 37060 55480 01671 31438 24360 00000
28	522 71176 41648 65720 29262 82073 27503 69684 37500
29	56 48350 56515 14793 95103 49161 76415 23464 06250
30	2 92156 06371 47316 92850 18060 09124 92592 96875

Stirling Π′ Numbers $n=1(1)16$

m	$\Pi_1'^{(m)}$	$\Pi_2'^{(m)}$	$\Pi_3'^{(m)}$	$\Pi_4'^{(m)}$	$\Pi_5'^{(m)}$	$\Pi_6'^{(m)}$	$\Pi_7'^{(m)}$	$\Pi_8'^{(m)}$	$\Pi_9'^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		3	10	25	56	119	246	501	1012
3			15	105	490	1918	6825	22935	74316
4				105	1260	9450	56980	3 02995	14 87200
5					945	17325	1 90575	16 36635	121 22110
6						10395	2 70270	40 99095	475 07460
7							1 35135	47 29725	945 94500
8								20 27025	918 91800
9									344 59425

m	$\Pi_{10}'^{(m)}$	$\Pi_{11}'^{(m)}$	$\Pi_{12}'^{(m)}$	$\Pi_{13}'^{(m)}$
1	1	1	1	1
2	2035	4082	8177	16368
3	2 35092	7 31731	22 52341	68 79678
4	69 14908	309 50920	1347 79645	5751 56036
5	814 31350	5108 80370	30496 16570	1 75393 36815
6	4668 76410	41041 60060	3 33099 26650	25 47526 58160
7	14222 80860	1 78928 64990	19 91249 36010	202 67631 58420
8	23432 40900	4 43469 82680	69 47402 96250	954 04210 90200
9	19641 87225	6 21992 62125	144 67330 12725	2761 01433 35775
10	6547 29075	4 58310 35250	176 44948 57125	4947 30748 51200
11		1 37493 10575	115 95251 91825	5333 81588 23950
12			31 62341 43225	3162 34143 22500
13				790 58535 80625

m	$\Pi_{14}'^{(m)}$	$\Pi_{15}'^{(m)}$	$\Pi_{16}'^{(m)}$
1	1	1	1
2	32751	65518	1 31053
3	209 00922	632 59533	1909 57923
4	24175 78670	1 00465 31276	4 13880 56231
5	9 80494 92723	53 61814 58345	288 18379 17245
6	186 17633 48445	1313 15699 45130	9004 92302 96025
7	1928 23952 72140	17407 37972 22325	1 50741 11193 49135
8	11888 53950 48420	1 37529 58563 74440	15 00606 41871 08995
9	45838 05540 06375	6 85978 90721 71035	94 74520 22025 25875
10	1 12976 40452 34825	22 29877 37485 01250	394 48246 42892 88375
11	1 77407 35434 92250	47 75891 01092 85375	1105 11612 86631 71625
12	1 71293 49424 68750	66 68113 14404 23500	2089 66415 02357 87125
13	92498 48689 33125	58 27404 67427 86875	2624 63428 79880 87375
14	21345 80466 76875	28 88798 89836 03750	2094 37920 13112 71875
15		6 19028 33536 29375	959 49391 98125 53125
16			191 89878 39625 10625

m	$\Pi'^{(m)}_{17}$	$\Pi'^{(m)}_{18}$
1	1	1
2	2 62124	5 24267
3	5753 63776	17313 33808
4	16 93713 83384	68 95681 72832
5	1527 83387 67076	8011 78642 69828
6	60369 58159 55540	3 97357 66873 75988
7	12 62301 01322 52520	102 84977 05087 00600
8	156 22638 03612 51200	1565 38629 61963 22600
9	1227 85842 45004 57750	15112 61170 98966 50950
10	6408 19990 01585 56500	97234 17646 30979 24250
11	22807 30395 11056 74000	4 30309 94066 66019 96000
12	56019 22140 53982 51000	13 33642 47144 68435 58000
13	94720 50610 06829 62500	29 11943 22147 08260 42500
14	1 08060 33745 80004 27500	44 49180 41353 31778 22500
15	79318 16403 78377 25000	46 47703 25922 35795 55000
16	33774 18597 74018 70000	31 57886 38888 70748 45000
17	6332 65987 07628 50625	12 55977 54103 46320 40625
18		2 21643 09547 66997 71875

m	$\Pi'^{(m)}_{19}$	$\Pi'^{(m)}_{20}$
1	1	1
2	10 48554	20 97129
3	52050 11031	1 56381 01281
4	279 63620 35104	1130 51633 94129
5	41644 93893 24276	2 14935 96354 63876
6	25 76428 88667 31800	164 99696 79334 97700
7	819 28781 07453 03900	6404 88618 57521 54100
8	15197 18440 27967 96400	1 43698 24611 24975 76500
9	1 78278 93538 63705 68750	20 30031 58175 56454 17950
10	13 95494 89250 80854 69100	191 25038 05128 56011 84750
11	75 53200 46476 24617 59250	1249 50051 88762 96434 24750
12	289 13007 87736 01825 76000	5811 05308 93595 85054 48750
13	791 98178 49397 28888 50500	19547 92572 49717 33974 88500
14	1554 70708 87653 09228 75000	47901 29814 57253 82523 16500
15	2165 38502 53494 85614 67500	85340 81639 82627 97997 62500
16	2085 48093 03579 49023 90000	1 09156 17077 29591 80896 02500
17	1318 77641 80863 63642 65625	97496 51260 03543 46785 55625
18	492 04767 19582 73493 56250	57651 58556 44443 77662 40625
19	82 00794 53263 78915 59375	20255 96249 56155 92151 65625
20		3198 30986 77287 77708 15625

Stirling Π' Numbers $n=21(1)23$

m	$\Pi'^{(m)}_{21}$	$\Pi'^{(m)}_{22}$
1	1	1
2	41 94280	83 88583
3	4 69625 37810	14 09882 76150
4	4559 59680 07260	18355 79354 74290
5	11 02942 72621 72605	56 33263 14790 51785
6	1045 86515 81230 46976	6572 98548 48169 42191
7	49289 12143 44695 16600	3 74308 07446 87319 31528
8	13 28922 78210 10409 26800	120 60766 77840 79433 95800
9	224 37533 37306 32384 80050	2418 05483 82060 03741 24450
10	2521 51327 96552 53743 86000	32170 76814 22021 41367 41550
11	19673 26750 35377 97143 99500	2 97094 36748 78838 88387 46500
12	1 09716 65367 63565 06549 77000	19 65817 67173 30253 84349 07500
13	4 45887 78637 34988 48471 59250	95 26907 44785 16062 52822 88250
14	13 35247 64868 91943 10470 40000	342 99539 60472 11800 43091 33750
15	29 56657 68107 43303 58275 15000	924 18780 56892 59505 51061 65000
16	48 18768 12270 48076 22250 90000	1864 96624 16302 71452 12194 95000
17	56 96219 03280 55135 88507 38125	2799 48912 22047 64206 50159 68125
18	47 42596 01897 34639 75774 45000	3075 19270 62093 73815 15727 96875
19	26 33275 12443 00269 79715 31250	2397 36068 12310 90716 45568 93750
20	8 74204 69717 91992 40229 37500	1254 48374 04521 50909 72915 31250
21	1 31130 70457 68798 86034 40625	394 70342 07764 08456 96356 28125
22		56 38620 29680 58350 99479 46875

m	$\Pi'^{(m)}_{23}$
1	1
2	167 77190
3	42 31745 43025
4	73789 74370 77060
5	286 61922 16530 64755
6	41015 22659 03151 03126
7	28 10773 10034 08148 44235
8	1077 15376 46132 55051 12240
9	25501 33124 51604 96123 89850
10	3 99085 43624 46135 33393 97900
11	43 29673 39105 93934 37386 82650
12	336 91020 55538 43568 17362 71000
13	1926 53415 33272 67697 38915 09750
14	8231 62222 58875 43457 04902 49500
15	26553 64673 90857 29198 60304 23750
16	64958 59648 22762 04443 35461 90000
17	1 20324 99850 10615 78143 28317 63125
18	1 67333 03359 99592 96932 89490 68750
19	1 71632 75389 79750 50034 10656 53125
20	1 25778 82342 07488 28285 72201 62500
21	62231 57267 57470 66714 58840 34375
22	18607 44697 94592 55828 28224 68750
23	2537 37913 35626 25794 76576 09375

m	$\Pi'^{(m)}_{24}$
1	1
2	335 54405
3	126 99598 36015
4	2 96301 54609 69915
5	1453 75723 65034 81455
6	2 54403 31696 98294 96651
7	209 05868 50009 51570 03425
8	9488 56977 80116 93010 69205
9	2 63980 90167 40686 26751 00330
10	48 32398 29353 64317 06028 44050
11	611 95312 13397 01879 46650 37750
12	5558 30815 35169 10521 16891 44750
13	37173 71139 31928 48520 30953 82750
14	1 86524 47483 55345 13202 08493 53750
15	7 11106 34567 00125 89346 90858 37250
16	20 74929 76654 07627 09839 19255 66250
17	46 43868 83380 90950 06169 99875 73125
18	79 45319 54334 27920 48666 71855 25625
19	102 89009 73525 98164 21829 61082 96875
20	98 95784 88602 79037 17181 02263 34375
21	68 41131 25670 36368 45578 12518 71875
22	32 09784 60395 67216 30378 68758 59375
23	9 14302 28112 70661 61380 59585 78125
24	1 19256 81927 74434 12353 99076 40625

m	$\Pi'^{(m)}_{25}$
1	1
2	671 08836
3	381 07854 76980
4	11 88762 07192 88080
5	7354 71363 08855 34810
6	15 70032 61891 40814 23556
7	1542 27582 32673 08134 20156
8	82598 43614 41239 94326 63240
9	26 88950 91774 10035 10111 86735
10	572 99333 59228 26503 69818 51720
11	8422 82373 74744 71771 24149 57000
12	88730 01021 04321 93914 82110 96000
13	6 88915 64979 16327 20047 27383 31500
14	40 23943 68063 88114 28600 95154 97000
15	179 41049 70363 60348 55084 94123 55000
16	616 43130 09145 27069 31303 42425 50000
17	1640 17890 60292 58862 08296 87369 59375
18	3380 58242 80015 22471 35140 88175 32500
19	5371 39925 33367 65701 07431 50352 42500
20	6506 32126 07198 99969 04123 32917 50000
21	5889 74076 26203 20410 30286 64743 56250
22	3853 07299 09541 51707 64924 88550 12500
23	1718 88828 85188 84383 39552 02126 87500
24	467 48673 15675 78176 42764 37951 25000
25	58 43584 14459 47272 05345 54743 90625

Stirling Π' Numbers $n=26(1)27$

m	$\Pi'^{(m)}_{26}$
1	1
2	1342 17699
3	1143 42354 78348
4	47 66099 56559 84740
5	37130 19677 60063 16450
6	96 48191 83604 19401 20446
7	11298 34120 09236 62994 94884
8	7 11682 59132 08131 23041 71068
9	270 08905 08856 92473 98112 30775
10	6671 06618 04376 16265 52100 52925
11	1 13278 82120 54409 43616 79111 88920
12	13 76404 60081 17417 82513 78865 61000
13	123 27643 83528 76487 29377 76199 57500
14	832 02921 87081 70360 82257 00118 86500
15	4300 73492 78009 29799 70312 18052 05000
16	17218 73119 31232 07399 59337 37873 55000
17	53773 15604 09075 37566 55790 67154 09375
18	1 31378 17666 32855 35553 89301 44048 38125
19	2 50802 21264 54655 37059 87397 36410 37500
20	3 71839 39161 45524 55929 16884 24209 12500
21	4 22975 33400 81421 27192 25692 73819 81250
22	3 61585 42164 41463 96852 51819 91050 18750
23	2 24481 93420 17336 22785 26090 99324 12500
24	95445 20769 50472 11020 64394 15046 87500
25	24835 23261 45275 90622 71857 66160 15625
26	2980 22791 37433 10874 72622 91939 21875

m	$\Pi'^{(m)}_{27}$
1	1
2	2684 35426
3	3430 65987 48315
4	190 98700 96882 89400
5	1 87128 47474 53671 09190
6	590 77317 31308 38428 49076
7	82272 29171 23594 81204 38906
8	60 77604 33139 79095 26161 94600
9	2679 89036 49335 16858 89470 64355
10	76433 86763 62610 91718 53048 37150
11	14 92896 48193 60421 81608 97950 36345
12	208 21450 41554 76572 47603 52639 10960
13	2139 39149 29039 73629 99948 66353 26500
14	16579 46659 60294 44543 26708 49647 11000
15	98624 22188 40489 31789 27219 75654 21500
16	4 56130 56605 76103 69981 02509 64162 90000
17	16 54549 09399 97260 56813 99948 70182 24375
18	47 30826 04573 90712 92898 62215 47650 98750
19	106 77259 99011 16943 04062 79114 73974 28125
20	189 73689 61398 24638 23337 57963 59059 75000
21	263 58933 42005 49500 99708 33106 88044 93750
22	282 57695 30856 20428 35983 73289 46455 12500
23	228 80770 14720 30467 69834 39268 45914 06250
24	135 14781 69476 78142 03758 50009 27331 25000
25	54 88586 40781 05975 27620 80543 21394 53125
26	13 68918 02171 27607 95124 24794 30747 81250
27	1 57952 07942 83954 76360 49014 72778 59375

m	$\Pi'^{(m)}_{28}$
1	1
2	5368 70881
3	10292 78493 07725
4	765 01154 33143 55365
5	9 41753 95803 68608 06750
6	3606 39143 54510 01716 97726
7	5 95992 32987 29648 74999 40926
8	515 00364 86111 58580 51449 18510
9	26306 95084 37048 99159 47065 84795
10	8 63494 61986 51510 40964 40897 52635
11	193 26348 27147 43854 83002 93292 11495
12	3080 80367 78207 75320 53992 51733 48975
13	36140 66957 39707 20089 03473 68156 82900
14	3 19827 58355 34751 42435 71814 15543 40500
15	21 75700 92529 39706 47656 30053 19991 84500
16	115 38930 59793 58699 86635 10603 79737 64500
17	481 97079 50453 01992 45003 09552 16265 74375
18	1596 09578 05329 09558 28805 17570 15918 74375
19	4204 85937 91611 94712 50529 65091 97456 76875
20	8813 05011 81489 89087 57702 77664 57986 21875
21	14642 74703 29231 22156 14078 77496 83811 68750
22	19132 57034 37105 74972 77350 34605 36214 68750
23	19391 42478 81377 22175 05377 67647 78779 68750
24	14912 74038 18178 29261 51758 02913 97567 18750
25	8399 83308 32319 12767 85962 14062 56088 28125
26	3264 86948 17849 34496 37133 13442 33353 28125
27	781 86279 31705 57607 98442 62290 25403 90625
28	86 87364 36856 17511 99826 95810 02822 65625

Stirling Π' Numbers $n = 29$

m	$\Pi'^{(m)}_{29}$
1	1
2	10737 41792
3	30880 01909 20486
4	3063 33986 44352 68660
5	47 34015 17111 36777 60795
6	21958 54495 84385 42976 15856
7	42 98170 00935 15391 85090 06892
8	4334 58642 76435 35999 11572 21416
9	2 55817 69259 19569 59914 27212 48025
10	96 34610 33071 22965 77703 97477 48560
11	2462 66121 16095 91309 10644 21216 79210
12	44700 18344 24390 58039 68027 52406 47500
13	5 96141 65525 22711 49299 58851 07111 85675
14	59 95494 29185 54222 37839 51292 80194 48800
15	463 88099 97220 89908 39580 38806 68244 09000
16	2803 53730 27990 86282 83038 92001 55443 50000
17	13386 02228 48412 75365 63632 39557 64711 66875
18	50900 38062 16762 63701 88635 55662 34761 60000
19	1 54908 82988 91094 48777 13906 62544 99859 56250
20	3 78094 25256 27171 27951 79478 77706 37649 27500
21	7 39337 14348 06860 30570 23090 32998 01370 15625
22	11 53053 89920 77887 57208 05646 36159 87307 50000
23	14 21763 85765 64069 33637 68554 20772 58881 87500
24	13 66259 85814 67894 55379 21831 87620 38156 25000
25	10 00371 06731 71427 70056 92229 06004 73267 96875
26	5 38477 59302 09315 86370 07417 08878 95952 50000
27	2 00678 11691 37764 52716 00273 21165 20335 93750
28	46216 77844 07485 16383 07941 70935 01653 12500
29	4951 79769 00801 98183 90136 61171 60891 40625

m	$\Pi'^{(m)}_{30}$
1	1
2	21474 83615
3	92643 49324 98802
4	12263 54986 40414 50678
5	237 74229 41095 91879 38415
6	1 33408 17506 05210 45073 22961
7	308 77697 68396 45618 42772 19060
8	36267 01432 46083 57491 40910 26332
9	24 67073 51757 80670 07194 84656 46033
10	1063 22993 31820 92872 13696 36061 58575
11	30943 11745 99904 23031 25245 32484 13710
12	6 37371 31098 52619 40149 52742 99726 17610
13	96 27249 22286 19653 78561 22219 96886 08775
14	1095 71011 26182 35707 49635 48695 28532 67225
15	9596 23248 42477 34410 58644 38983 52218 82200
16	65731 24183 22794 26403 09740 18325 58080 05000
17	3 56525 09477 10596 50226 01541 04551 50499 36875
18	15 45349 89857 77126 88818 86162 61131 67157 23125
19	53 86486 03773 35401 84456 18732 60147 65888 48750
20	151 52417 71582 07055 49115 71000 18832 46104 06250
21	344 30792 64123 02630 39564 58835 78277 11237 03125
22	630 73380 10008 63402 17659 01826 78415 90642 96875
23	926 59371 48490 23748 48485 70357 58082 94273 12500
24	1081 43721 05131 25144 11898 57338 03836 36489 37500
25	987 87309 02285 51998 41900 84647 81618 92136 71875
26	690 20826 12098 70735 98752 65442 61113 24503 28125
27	355 73054 36584 41330 60056 22733 68682 22410 31250
28	127 32722 46042 62162 63538 37940 92597 05435 93750
29	28 24175 28257 57396 97551 74580 88207 61732 03125
30	2 92156 06371 47316 92850 18060 09124 92592 96875

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The problem of characterizing the class of *D*-stable matrices has remained unsolved since its suggestion in a paper by Arrow and McManus in 1958. In this note we present a necessary condition involving real quadratic forms which is not sufficient and a sufficient condition involving complex quadratic forms which is not necessary.

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Denote the real part $A \in M_n(C)$ by $H(A) = 1/2(A + A^*)$. We provide dual inequalities relating $H(A^{-1})$ and $H(A)^{-1}$ and an identity between two functions of A when A satisfies $H(A) > 0$. As an application we give an inequality (for matrices A satisfying $H(A) > 0$) which generalizes Hadamard's determinantal inequality for positive definite matrices.

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Coefficient-wise multiplication was introduced by Hadamard and has been studied for certain square matrices by I. Schur and later authors. For $A \epsilon M_n(C)$, the n by n complex matrices, this paper examines the Hadamard product of A and A^*. Upper estimates are given for the largest characteristic root of this necessarily Hermitian product, and three conditions on A sufficient for the product to be positive definite are presented.

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Let V be a complex inner product space of dimension n and let $\otimes^m V$ be the space of m contravariant tensors over V. Given a subgroup G of S_m and an irreducible character χ on G, we define a subspace $V_\chi^m(G)$ in $\otimes^m V$. If $T: V \to V$ is linear, let $\Pi \cdot T: \otimes^m V \to \otimes^m V$ be the mth Kronecker product of T. Then $V_\chi^m(G)$ is invariant under $\Pi \cdot T$ and we let $K(T)$ be the restriction of $\Pi \cdot T$ to $V_\chi^m(G)$. In this paper we prove that if the rank of T is large enough, then the elementary divisors of $K(T)$ are linear if and only if the elementary divisors of T are linear. This result has previously been proved only for the case that χ is linear.

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Key words: approximation; asymptotics; asymptotic theory; error analysis; ordinary differential equations; turning points.

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